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SOME REMARKS ON ADDITIVE ARITHMETICAL FUNCTIONS

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1. We call that a function $f(n)$ defined on the set of natural numbers is restrictedly additive (or simply additive), if $f(nm)=f(n)+f(m)$ holds for all coprime m, n Furthermore, we say, that $f(n)$ is completely additive, if $f(nm)=f(n)+f(m)$ holds for all pairs m, n of integers.

The letters $p, p_1, p_2, \ldots; q, q_1, \ldots; q'$ stand for prime numbers. $c_1, c_1, c_2, \ldots$; δ, K denote suitable positive constants, not necessarily the same at every occurence.

2. In the paper [1] I proved the following analogon of the Erdős-Wintner theorem: if $f(n)$ is an additive function satisfying the conditions

a) $\sum\limits_{|f(p)|<1} \frac{f(p)}{p}$ converges,

b) $\sum\limits_{|f(p)|<1} \frac{f^2(p)}{p}<\infty$,

c) $\sum\limits_{|f(p)|\geq 1} \frac{1}{p}<\infty$,

then $f(p+1)$ has a limit distribution, i. e. there exists a distribution function $F(\alpha)$, such that

$$\lim_{x\to\infty} (\mathrm{li}\, x)^{-1} N\{p\leq x,\ f(p+1)<\alpha\}=F(\alpha) \tag{2.1}$$

holds for all continuity points of its.

The question, whether the conditions a), b), c) are necessary for (2. 1) was suggested to me by Prof. Kubilius and Prof. Erdős.

I think that the answer is affirmative. However presently I can prove only the following weaker assertion.

Theorem 1. *Let $f(n)$ be an additive arithmetical function, for which $f(p)$ is bounded. Assuming that for a suitable distribution function $F(\alpha)$ the relation* (2.1) *holds for all continuity points of it, the conditions* a), b), c) *are satisfied.*

Proof. Let

$$A(x)=\sum_{q<x} \frac{f(q)}{q}, \quad B(x)=\Big(\sum_{q<x} \frac{f^2(q)}{q}\Big)^{\frac{1}{2}}. \tag{2.2}$$

First we prove, that $B(x)$ is bounded for $x\to\infty$. Hence the conditions b), c) will immediately follow. Suppose the contrary, $B(x)\to\infty$. Then using the large sieve repeating the arguments of Barban, A. I. Vinogradov and Levin in [2] we deduce:

$$N\Big\{p\leq x, p+1 \text{ square free}, \Big(f(p+1)-A(x)\Big)\Big(B(x)\Big)^{-1}>\alpha\Big\}(C\,\mathrm{li}\, x)^{-1}\to\Phi(\alpha) \tag{2.3}$$

for all real α as $x\to\infty$, where

$$\Phi(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}}\,du,\quad C=\prod_p\left(1-\frac{1}{p(p-1)}\right).$$

Hence it follows, that $f(p+1)>A(x)+B(x)$ for at least $\delta\,\mathrm{li}\,x$ of primes $p\leqslant x$ and $f(p+1)<A(x)-B(x)$ for at least $\delta\,\mathrm{li}\,x$ of primes $p\leqslant x$, with a suitable constant $\delta\,(>0)$. This contradicts to (2.1).

Now we prove, that $A(x)$ is bounded. Using the large sieve in the form due to Barban, say, we easily deduce the following Turán-type inequality:

$$\sum_{\substack{p\leqslant x\\ p+1-\text{square free}}}\left(f(p+1)-A(x)\right)^2\ll \mathrm{li}\,x. \tag{2.4}$$

Assuming, that $\limsup_{x\to\infty}|A(x)|=\infty$, from (2.4) follows the existence of two sequences x_ν, K_ν tending to infinity, such that $|f(p+1)|>K_\nu$ for at least $\delta\,\mathrm{li}\,x_\nu$ of primes $p\leqslant x$, with a positive constant δ. This contradicts to (2.1).

Let

$$(-\infty<)a=\varliminf_{x\to\infty}A(x)\leqslant\varlimsup_{x\to\infty}A(x)=A\,(<+\infty). \tag{2.5}$$

Now we prove, that $A=a$, which completes the proof of Theorem 1. I proved in [3] that for an additive function $f(n)$ for which $f(p)$ and $B(x)$ are bounded there exists a distribution function $G(\alpha)$, such that

$$\lim_{x\to\infty}(\mathrm{li}\,x)^{-1}N\{p\leqslant x,\ f(p+1)-A(x)<\alpha\}=G(\alpha) \tag{2.6}$$

in all continuity points of it.

Observing (2.1) and (2.6) we obtain that $G(\alpha+A-\varepsilon)\leqslant G(\alpha+a+\varepsilon)$ for all α and all $\varepsilon>0$. Hence $a=A$ and thus Theorem 1 holds.

3. Let $f(n)$ be a completely additive function, satisfying the inequality

$$|f(p+1)|\leqslant C\log(p+1) \tag{3.1}$$

for all primes p.

I believe, that from (3.1)

$$|f(n)|\leqslant AC\log n\qquad (n=1,\ 2,\ \ldots)$$

follows with an absolute constant A. But I am unable to prove this, presently.

Now we prove a weaker result on the assumption of the extended Riemann-hypothesis stating that all the Dirichlet L-functions have no zeros in the halfplane $\mathrm{Re}\,s>\frac{1}{2}$.

Theorem 2. *Suppose that the extended Riemann-hypothesis is true. If $f(n)$ is a completely additive function satisfying* (3.1) *then*

$$|f(n)|\leqslant K(\log n)(\log\log 10n)\qquad (n=1,\ 2,\ \ldots), \tag{3.2}$$

where K is a suitable constant dependig only on C and on the values $f(p)$ for $p\leqslant c_1$.

Previously I showed, that there exists a suitable constant K, such that, if $f(q)=0$ for all $q\leqslant K$ and $f(p+1)=0$ for all primes p, then $f(n)=0$, identically (See [4]).

Proof. From the extended Riemann-hypothesis

$$\pi(x,\ q,\ -1)=\frac{\operatorname{li} x}{q-1}+O\left(x^{\frac{1}{2}}\log x\right)$$

follows (see [5], p. 251). Hence

$$\pi(x_q,\ q,\ -1)>0{,}9\ \frac{\operatorname{li} x_q}{q},\quad x_q=q^2(\log q)^5$$

for all $q\geqslant c_1$.

By the Brun sieve we can deduce the following assertion. There exist positive constants δ ($<0{,}01$) and c_2 such that for all $q>c_2$ can be found a prime $p\leqslant x_q-1$ satisfying the conditions: $p+1\equiv 0 \pmod q$, $p+1=qm$, all prime factors of m are smaller than $q^{1-\delta}$. Such an m can be written as $m=m_1m_2$, where $m_1<q^{1-\delta}$, $m_2<q^{1-\delta}$.

Thus by such a p from (3.1)

$$|f(q)|\leqslant|f(p+1)|+|f(m)|\leqslant C\log x_q+|f(m_1)|+|f(m_2)| \tag{3.4}$$

follows.

Let

$$E(x)=(\log x)\,H(x),\quad H(x)=K\log\log 10\,x, \tag{3.5}$$

K be a sufficiently large constant.

Let c_3 be such a large constant that for all $q\geqslant c_3$.

$$5\ \frac{\log\log q}{\log q}<0{,}001\log\frac{1}{1-\delta},\quad \log\log 10\,q^{1-\delta}<\log\log 10\,q-\frac{1}{2}\log\frac{1}{1-\delta},$$

$$x_q<q^3.$$

Let K be such a large constant, that

$$|f(q')|\leqslant E(q') \tag{3.7}$$

for all primes $q'\leqslant \max(c_1,\ c_2,\ c_3)=c_4$.

Now we deduce by induction, that (3.7) holds for all integers. Assuming, that (3.7) holds for all primes $q'\leqslant X$ we deduce

$$|f(n)|\leqslant E(n) \tag{3.8}$$

for all integers $n\leqslant X$. This is an easy consequence of the inequality $E(x^A)\leqslant AE(x)$ holding for $x\leqslant 1$, $A\leqslant 1$.

Let $q>c_4$ and suppose that (3.7) holds for all $q'<q$. Consequently (3.8) holds for all $n<q$. Using (3.4) and taking into account (3.6) we have by using easy calculation, that

$$|f(q)|\leqslant 3\,C\log q+(\log m_1)\,H(m_1)+(\log m_2)\,H(m_2)\leqslant 3\,C\log q+$$
$$+(\log m)\,H(q^{1-\delta})\leqslant 3\,C\log q+(\log q+5\log\log q)\,K\log\log 10q^{1-\delta}\leqslant E(q).$$

Thus (3.7) holds for $q'=q$. Hence Theorem 2 by induction follows.

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References

1. I. Kátai, On distribution of arithmetical functions on the set of prime plus one, Compositio Math (in print).
2. М. Б. Барбан, А. И. Виноградов, Б. В. Левин, Предельные законы для функций класса H И. П. Кубилюса, заданных на множестве „сдвинутых" простых чисел, Лит. матем. сб., V, № 1 (1965), 1–8.

3. I. Kátai. On the distribution of artithmetical functions, Acta Math. Acad. Sci. Hung (in print).
4. I. Kátai, On sets characterizing number-theoretical I–II, Acta Arithmetica, I.: 13 (1968), 315–320; II (in print).
5. K. Prachar, Primzahlverteilung, Springer Verlag, 1957.

PASTABOS APIE ADITYVINES ARITMETINES FUNKCIJAS

I. Katajus

(*Reziumė*)

Straipsnyje [1] buvo įrodytas šitoks Erdiošo – Vintnerio teoremos analogas.

Sakykime, $f(n)$ – adityvinė funkcija, kuriai

$$\text{a)} \sum_{|f(p)|<1} \frac{f(p)}{p} \text{ konverguoja, b)} \sum_{|f(p)|<1} \frac{f^2(p)}{p} < \infty, \text{ c)} \sum_{|f(p)|\geq 1} \frac{1}{p} < \infty$$

(p – pirminis skaičius). Tada $f(p+1)$ turi ribinį pasiskirstymą.

Šiame straipsnyje nagrinėjamas sąlygų būtinumas.

1. teorema. *Tarkime, kad $f(n)$ – aprėžta adityvinė funkcija. Tam, kad $f(p+1)$ turėtų ribinį pasiskirstymą, sąlygos* a, b *ir* c *būtinos.*

2 teorema. *Tarkime, kad $f(n)$ – pilnai adityvinė funkcija, kuriai galioja nelygybė $|f(p+1)| \leq C\log(p+1)$, ir Rimano – Pilco hipotezė yra teisinga.*

Tada

$$|f(n)| \leq K_f (\log n)(\log\log 10n) \quad (n=1, 2, \ldots);$$

Čia K_f – konstanta, priklausanti nuo f.

НЕКОТОРЫЕ ЗАМЕЧАНИЯ К АДДИТИВНЫМ АРИФМЕТИЧЕСКИМ ФУНКЦИЯМ

И. Катаи

(*Резюме*)

В работе [1] доказывался следующий аналог теоремы Эрдеша – Винтнера. Пусть $f(n)$ – аддитивная функция, для которой

$$\text{а)} \sum_{|f(p)|<1} \frac{f(p)}{p} \text{ сходится, б)} \sum_{|f(p)|<1} \frac{f^2(p)}{p} < \infty, \text{ в)} \sum_{|f(p)|\geq 1} \frac{1}{p} < \infty.$$

Здесь и позже p пробегает простые числа.) Тогда $f(p+1)$ имеет предельное распределен ие.

В этой заметке изучается необходимость этих условий.

Теорема 1. *Пусть $f(n)$ – ограниченная аддитивная функция. Для того чтобы $f(p+1)$ имела предельное распределение, условия* а), б) *и* в) *необходимы.*

Теорема 2. *Пусть $f(n)$ – тотально-аддитивная функция, для которой имеет место неравенство $|f(p+1)| \leq C\log(p+1)$. Предполагая справедливость гипотезы Римана – Пильца, доказывается, что $|f(n)| \leq K_f(\log n)(\log\log 10n)$ для всех* $n=1, 2, \ldots,$ где K_f-некоторая (зависящая от f) постоянная.