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SOME REMARKS ON THE RAREFACTION OF THE RENEWAL PROCESSES

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1. We start from the following rather obvious

Theorem 1. *Let $\xi_1, \xi_2, \ldots$ be a sequence of independent and identically distributed random variables with finite mean-value μ. Let further ν_n be a sequence of positive integer-valued random variables, which for each n is independent of the sequence $\{\xi_i\}$. Assume that there exists a sequence $\omega(n)$ of positive numbers such that $\omega(n)\to+\infty$ as $n\to+\infty$ and*

$$\lim_{n\to+\infty} P\left(\frac{\nu_n}{\omega(n)}<x\right)=G(x),$$

where $G(x)$ is a distribution function. Then

$$\lim_{n\to+\infty} P\left(\frac{\xi_1+\ldots+\xi_{\nu_n}}{\omega(n)}<x\right)=\begin{cases} G\left(\frac{x}{\mu}\right), & \text{if } \mu>0,\\ E(x), & \text{if } \mu=0\\ 1-G\left(\frac{x}{\mu}\right), & \text{if } \mu<0,\end{cases}$$

where

$$E(x)=\begin{cases} 0, & \text{if } x\leqq 0\\ 1, & \text{if } x>0.\end{cases}$$

Proof. We prove first that the random variables

$$\zeta_n=\frac{\xi_1+\ldots+\xi_{\nu_n}}{\nu_n} \qquad (n=1,\ 2,\ \ldots)$$

converge in probability to μ as $n\to+\infty$. In fact, by the total probability theorem the characteristic function of ζ_n is

$$M\left(e^{it\zeta_n}\right)=M\left(f\left(\frac{t}{\nu_n}\right)^{\nu_n}\right),$$

where $f(t)$ is the characteristic function of ξ_i $(i=1, 2, \ldots)$. Now, by the supposition, ν_n converges in probability to $+\infty$ as $n\to+\infty$ and so $f\left(\frac{t}{\nu_n}\right)^{\nu_n}$ converges in probability to $e^{i\mu t}$. We have further obvioiusly $\left|f\left(\frac{t}{\nu_n}\right)^{\nu_n}\right|\leqq 1$. Thus, if we apply the Lebesgue's convergence theorem, it follows that ζ_n converges in probability to μ. Now let us consider the random variable

$$\eta_n=\zeta_n\,\frac{\nu_n}{\omega(n)}\,.$$

By a theorem of H. Cramer [9] the distribution function of η_n converges to $G\left(\frac{x}{\mu}\right)$, $E(x)$ or $1-G\left(\frac{x}{\mu}\right)$ according to the sign of μ.

In this paper we shall be concerned with the following rarefaction models for renewal processes.

Let $\tau_0 \equiv 0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots$ be a renewal process, i.e. let the non-negative random variables $\tau_i - \tau_{i-1}$ $(i=1, 2, \ldots)$ be independent and identically distributed with common distribution function $F(x)$. In the sequel we suppose that the mean-value μ of the random variables $\xi_i = \tau_i - \tau_{i-1}$ is finite and positive.

We define **the general rarefaction model** as follows: let $Z_1^{(n)}, Z_2^{(n)}, \ldots; (n=1, 2, \ldots)$ be a sequence of positive integer random variables which are independent for fixed n, and also independent of the renewal process $\tau_0 \equiv 0 \leqslant \tau_i \leqslant \ldots$. Suppose further that $Z_i^{(n)}$ $(i=1, 2, \ldots)$ are for fixed n indentically distributed. Put

$$\tau_0^{(n)} \equiv 0,$$

$$\tau_i^{(n)} = \tau_{Z_1^{(n)}+Z_2^{(n)}+\ldots+Z_i^{(n)}}. \qquad (i=1, 2, \ldots; n=1, 2, \ldots)$$

The process $\tau_0^{(n)} \equiv 0 \leqq \tau_1^{(n)} \leqslant \tau_2^{(n)} \leqslant \ldots$ will be called a rarefaction of the original renewal process.

It is obvious that the rarefaction is also a renewal process and the distribution function of the differences

$$\tau_i^{(n)} - \tau_{i-1}^{(n)} = \sum_{j=Z_1^{(n)}+\ldots+Z_{i-1}^{(n)}+1}^{Z_1^{(n)}+\ldots+Z_i^{(n)}} \xi_j \tag{1}$$

is the following:

$$\sum_{k=1}^{\infty} P(Z_i^{(n)}=k)\, F_{(x)}^{(*k)}, \tag{2}$$

where $F^{(*k)}(x)$ denotes the k-fold convolution of the distribution function $F(x)$ with itself. Moreover, if the mean-value of the random variables $Z_i^{(n)}$ exists, i.e.

$$M(Z_i^{(n)}) = M_n < +\infty,$$

then obviously

$$M(\tau_i^{(n)} - \tau_{i-1}^{(n)}) = \mu M_n. \qquad (i=1, 2, \ldots) \tag{3}$$

On the basis of Theorem 1. we prove now for this model

Theorem 2. *Suppose that there exists a sequence $\omega(n)$ of positive numbers such that $\omega(n) \to +\infty$ as $n \to +\infty$ and*

$$\lim_{n\to+\infty} P\left(\frac{Z_i^{(n)}}{\omega(n)} < x\right) = G(x), \qquad (i=1, 2, \ldots).$$

Then the limiting distribution of the random variables

$$\frac{\tau_i^{(n)} - \tau_{i-1}^{(n)}}{\mu\omega(n)} \tag{4}$$

is also $G(x)$. Moreover, every distribution, which is concentrated on the positive axis, is a possible limiting distribution for (4).

Proof. By (1) the random variable $\tau_i^{(n)}-\tau_{i-1}^{(n)}$ is a sum of a random number of independent and identically distributed random variables where the number of summands is $Z_i^{(n)}$.

New by Theorem 1. our first assertion follows.

Let $G(x)$ be a distribution function such that the corresponding probability mass is concentrated on the positive axis. Put an infinite sequence z_i of independent and identically distributed random variables with the distribution function $G(x)$ and define

$$Z_i^{(n)}=[nz_i]. \qquad (i=1,\ 2,\ \dots;\ n=1,\ 2,\ \dots)$$

Then we have obviously

$$\lim_{n\to+\infty} P\left(\frac{Z_i^{(n)}}{n}<x\right)=G(x).$$

If the random variables z_i are independent of the original renewal process $\tau_0\equiv 0\leqq \leqq\tau_1\leqq\tau_2\leqq\dots$, then the corresponding rarefied process $\tau_0^{(n)}\equiv 0\leqslant\tau_1^{(n)}\leqslant\tau_2^{(n)}\leqslant\dots$ is such that

$$\lim_{n\to+\infty} P\left(\frac{\tau_i^{(n)}-\tau_{i-1}^{(n)}}{\mu n}<x\right)=G(x). \qquad (i=1,\ 2,\ \dots)$$

This proves the second assertion.

Suppose now that $M(Z_i^{(n)})=M_n$ exists and $M_n\to+\infty$, further

$$\lim_{n\to+\infty} P\left(\frac{Z_i^{(n)}}{M_n}<x\right)=G(x),$$

where $G(x)$ is a distribution function. This enables us to imaginate the following renewal process:

$$t_0^{(n)}\equiv 0 \tag{5}$$

and for $i=1, 2, \dots$ the random variables $t_i^{(n)}$ are determined by the relations

$$t_i^{(n)}-t_{i-1}^{(n)}=\frac{\tau_i^{(n)}-\tau_{i-1}^{(n)}}{M_n}. \qquad (i=1,\ 2,\ \dots) \tag{6}$$

For fixed n these random variables are equally distributed and **independent.** Moreover by (3)

$$M(t_i^{(n)}-t_{i-1}^{(n)})=\mu. \qquad (i=1,\ 2,\ \dots)$$

If we rarefy the original renewal process according to the random variables $Z_i^{(n)}$ and then compress the new process $\tau_0^{(n)}\equiv 0\leqq\tau_1^{(n)}\leqq\tau_2^{(n)}\leqq\dots$ such that in the new scale the mean-value between consecutive renewal points be μ, then we obtain the process defined by (5) and (6).

Theorem 2. expresses the fact that the asymptotic behaviour in distribution of the rarefied renewal process does not depend on the stochastic behaviour of the or i-ginal renewal process. It depends only on the rarefying random variables.

2. A more **special rarefaction model** for renewal processes is the following: let $\nu_i^{(n)}$ $(i, n=1, 2, \dots)$ be a double entry table of independent and identically distributed random variables, which are independent of the original renewal process, and which take on the values $1, 2, \dots$. We suppose that their mean value M exists and $M>1$.

The rarefaction and compression will be made now step by step. In the first step we use the random variables $\nu_i^{(1)}$ $(i=1, 2, \dots)$ and we define $\tau_0^{(1)} \equiv 0$, further

$$\tau_i^{(1)} = \tau_{\nu_1^{(1)}+\nu_2^{(1)}+\dots+\nu_i^{(1)}}. \qquad (i=1,\ 2,\ \dots)$$

Then, obviously, the random variables

$$\tau_i^{(1)} - \tau_{i-1}^{(1)} \qquad (i=1,\ 2,\ \dots)$$

are independent and identically distributed with common distribution function

$$\sum_{k=1}^{\infty} P(\nu_i^{(1)}=k)\, F^{(*k)}(x).$$

The mathematical expectation of $\tau_i^{(1)} - \tau_{i-1}^{(1)}$ is μM, where M is the mean-value of $\nu_i^{(n)}$. The compression will be made in the following way: we put $t_0^{(1)} \equiv 0$ and define $t_i^{(1)}$ by the relations

$$t_i^{(1)} - t_{i-1}^{(1)} = \frac{\tau_i^{(1)} - \tau_{i-1}^{(1)}}{M}. \qquad (i=1,\ 2,\ \dots)$$

Thus, after rarefaction and compression we obtain in the first step the renewal process $t_0^{(1)} \equiv 0 \leqq t_1^{(1)} \leqq t_2^{(1)} \leqq \dots$ which has the mean-value μ.

The second step will be made now with the help of the random variables $\nu_i^{(2)}$, but now we rarefy and compress the renewal process $t_0^{(1)} \leqslant 0 \leqq t_1^{(1)} \leqq t_2^{(1)} \leqq \dots$. We obtain a new process $t_0^{(2)} \equiv 0 \leqq t_1^{(2)} \leqq t_2^{(2)} \leqq \dots$, the mean-value of which is also μ.

If the k-th step is already made, the $(k+1)$-th one is as follows: let $\tau_0^{(k+1)} \equiv 0$ and

$$\tau_i^{(k+1)} = t^{(k)}_{\nu_1^{(k+1)}+\nu_2^{(k+1)}+\dots\nu_i^{(k+1)}}. \qquad (i=1,\ 2,\ \dots)$$

Then, as it is easily seen, the differences

$$\tau_i^{(k+1)} - \tau_{i-1}^{(k+1)} \qquad (i=1,\ 2,\ \dots)$$

are independent and identically distributed non-negative random variables with mean-value μM. The $(k+1)$-th step is finished with the definition of $t_i^{(k+1)}$: let $t_0^{(k+1)} \equiv 0$ and let $t_i^{(k+1)}$ $(i=1, 2, \dots)$ be determined by the relations

$$t_i^{(k+1)} - t_{i-1}^{(k+1)} = \frac{\tau_i^{(k+1)} - \tau_{i-1}^{(k+1)}}{M}. \qquad (i=1,\ 2,\ \dots)$$

This rarefaction and compression procedure generalizes that of Renyi [2] and of others [7], because these authors consider only the case when the random variables $\nu_i^{(n)}$ have geometric distribution.

The renewal process $t_0^{(n)} \equiv 0 \leqq t_1^{(n)} \leqq t_2^{(n)} \leqq \dots$ can be expressed by the original process $\tau_0 \equiv 0 \leqq \tau_1 \leqslant \tau_2 \leqslant \dots$ in the following manner: let $Z_i^{(1)} = \nu_i^{(1)}$ $(i=1, 2, \dots)$ and define recursively the random variables $Z_i^{(n)}$ as follows:

$$Z_1^{(n)} = \sum_{i=1}^{\nu_1^{(n)}} Z_i^{(n-1)}, \quad Z_2^{(n)} = \sum_{i=\nu_1^{(n)}+1}^{\nu_1^{(n)}+\nu_2^{(n)}} Z_i^{(n-1)}, \quad \dots \qquad (n \geqq 2). \tag{8}$$

Then, putting

$$\tau_i^{(n)*} = \tau_{Z_1^{(n)}+Z_2^{(n)}+\dots+Z_i^{(n)}} \qquad (i=1,\ 2,\ \dots) \tag{9}$$

we see that $t_0^{(n)} \equiv 0$ and

$$t_i^{(n)} - t_{i-1}^{(n)} = \frac{\tau_i^{(n)*} - \tau_{i-1}^{(n)*}}{M^n}. \tag{10}$$

Also it can be easily seen that $\tau_i^{(n)*} - \tau_{i-1}^{(n)*}$ $(i=1, 2, \dots)$ is a sum of a random number of the independent random variables $\xi_j = \tau_j - \tau_{j-1}$, where the number of the summands is $Z_i^{(n)}$, i.e.

$$t_i^{(n)} - t_{i-1}^{(n)} = \sum_{j=\sum_{l=1}^{i-1} Z_l^{(n)}+1}^{\sum_{l=1}^{i} Z_l^{(n)}} \xi_j / M^n. \tag{11}$$

Let $f(s)$ denote the generating function of the random variables $\nu_i^{(n)}$; i.e.

$$f(s) = \sum_{k=1}^{\infty} P(\nu_i^{(n)} = k) s^k, \qquad (|s| \leq 1;\ i,\ n = 1,\ 2,\ \dots).$$

Then the independence of $\nu_i^{(n)}$ for every i and n, and the construction of the random variables $Z_i^{(n)}$ implies that the generating function of $Z_i^{(n)}$ is the n-fold iterate of the generating function $f(s)$ with itself:

$$f_n(s) = f\Bigg(f\Big(\dots\big(f(s)\big)\dots\Big)\Bigg).$$

Lemma. *Let us suppose that* $0 < D^2(\nu_i^{(n)}) < +\infty$ *and let* $M(\nu_i^{(n)}) = M$. *Then*

$$\lim_{n \to +\infty} P(Z_i^{(n)} < M^n x) = G(x), \tag{12}$$

where $G(x)$ *is a distribution function with mean-value* 1 *and dispersion*

$$D^2(\nu_i^{(n)})/(M^2 - M).$$

Moreover, $G(x)$ *is continuous. The class of the possible limit distributions coincides with the class of the possible limit distributions for Galton-Watson processes with mean-value* $M > 1$.

Proof. It is obvious that $M > 1$. Let us consider that Galton—Watson process (cf. [10]. Theorem 8.1.), in which the distribution of the number of the first generation is given by $f(s)$. Then as it is well-known, the generating function of the number of the n-th generation is $f_n(s)$. It is also known that under our assumptions the number of the n-th generation divided by M^n converges with probability 1 to a random variable W, the distribution function of which is continuous. Now, since the distribution function of $Z_i^{(n)}/M^n$ is the same as that of the n-th generation of the Galton—Watson process divided by M^n, we see that

$$\lim_{n \to +\infty} P(Z_i^{(n)} < xM^n) = G(x)$$

exists, where $G(x)$ is the distribution function of the random variable W. This proves our assertion.

This lemma enables us to prove

Theorem 3. *In the special rarefaction model* (10) *we have*

$$\lim_{n\to+\infty} P(t_i^{(n)}-t_{i-1}^{(n)}<\mu x)=G(x), \qquad (i=1,\ 2,\ \dots)$$

were $G(x)$ *is defined in the Lemma.*

Proof. By (11)

$$t_i^{(n)}-t_{i-1}^{(n)}=\frac{\tau_i^{(n)*}-\tau_{i-1}^{(n)*}}{M^n}=\sum_{j=\sum_{l=1}^{i-1} Z_l^{(n)}+1}^{\sum_{l=1}^{i} Z_l^{(n)}} \xi_j/M^n.$$

So

$$P(t_i^{(n)}-t_{i-1}^{(n)}<\mu x)=P\left(\left(\sum_{j=\sum_{l=1}^{i-1} Z_l^{(n)}+1}^{\sum_{l=1}^{i} Z_l^{(n)}} \xi_j/Z_i^{(n)}\right)\left(\frac{Z_i^{(n)}}{M^n}\right)<\mu x\right). \tag{13}$$

Applying Theorem 1. and the Lemma our assertion immediately follows.

3. Now we turn to the case where we don't suppose that the original renewal process $\tau_0\equiv 0\leqslant\tau_1\leqslant\tau_2\leqslant\dots$ has finite mean-value μ. In this case we give a necessary and sufficient condition to ensure that for a suitable choice of the positive constants δ_n ($\delta_n\to 0$, as $n\to+\infty$) the distribution function of the random variables

$$\delta_n(\tau_i^{(n)*}-\tau_{i-1}^{(n)*}) \qquad (i=1,\ 2,\ \dots)$$

converges to a limiting distribution as $n\to+\infty$. Here $\tau_i^{(n)}$ ($i, n=1, 2, \dots$) is defined by (9). Let us denote by $F(x)$ the distribution function and by $\varphi(s)$ the Laplace transform of the random variables $\tau_i-\tau_{i-1}$ ($i=1, 2, \dots$). We prove

Theorem 4. *Let us consider the special rarefaction model defined by* (8) *and* (9). *Suppose that the positive integer random variables* $\nu_i^{(n)}$ ($i, n=1, 2,\dots$) *are independent and identically distributed with finite dispersion* $D^2(\nu_i^{(n)})$, *and independent also of the original process. In order that for a suitable choice of the norming constants* $\delta_n>0$ ($n=1, 2, \dots$) *the random variables*

$$\delta_n(\tau_i^{(n)*}-\tau_{i-1}^{(n)*}) \tag{14}$$

($i=1, 2, \dots$ fixed; $n\to+\infty$) *have a limiting distribution, it is necessary and sufficient that one of the limits*

$$\lim_{n\to+\infty} M^n\left(1-\varphi(\delta_n s)\right)=0,$$

or

$$\lim_{n\to+\infty} M^n\left(1-\varphi(\delta_n s)\right)=Cs^\alpha$$

exist for $s>0$. *Here C is a positive constant and* $0<\alpha\leqslant 1$.

Proof. Necessity. Let us consider for $s>0$ the Laplace transform $\varphi_n(s)$ of (14). Then it can be easily seen, that

$$\varphi_n(s)=f_n\left(\varphi(\delta_n s)\right), \tag{15}$$

where $f_n(z)$ denotes the n-th iterate of the generating function $f(z)$ considered in Section 2. Now for $0<z\leqq 1$ $f(z)$ has an inverse $u(z)$ and the inverse function of $f_n(z)$ is $u_n(z)$ where $u_n(z)$ denotes the n-th iterate of $u(z)$. For $0<z\leqq 1$ the unique fixed point of $u(z)$ is 1. Further $u(z)$ satisfies all the conditions, which are necessary to ensure that the limit

$$\chi(z)=\lim_{n\to+\infty} M^n\Big(u_n(z)-1\Big)$$

exist. (See for example Kuczma [1], definitions: page 19,20, assertion: page 137). $\chi(z)$ is the solution of the so called Schröder equation

$$\chi\Big(u(z)\Big)=\frac{1}{M}\,\chi(z).$$

$\chi(z)$ is strictly monotonically increasing in (0, 1], twice differentiable, $\chi(1)=0$ and $\chi'(1)=1$.

Now from (15)

$$u_n\Big(\varphi_n(s)\Big)=\varphi(\delta_n s)\qquad (s>0)$$

and this implies

$$M^n\Big(u_n\Big(\varphi_n(s)\Big)-1\Big)=M^n\Big(\varphi(\delta_n s)-1\Big)\cdot \tag{16}$$

Now by supposition $\varphi_n(s)$ converges to the Laplace transform $h(s)$ of the limiting distribution. Further

$$\lim_{n\to+\infty}\Big(u_n\Big(h(s)\Big)-1\Big)\,M^n=\chi\Big(h(s)\Big)\cdot$$

Thus by the continuity of $\chi(z)$ and $h(s)$ it follows that

$$\lim_{n\to+\infty}\Big(u_n\Big(\varphi_n(s)\Big)-1\Big)\,M^n=\chi\Big(h(s)\Big),$$

which by (16) means that

$$\lim_{n\to+\infty}\Big(\varphi(\delta_n s)-1\Big)\,M^n=\chi\Big(h(s)\Big),\qquad (s>0).$$

We must now distinguish two cases. In the first one we consider the possibility $h(s)\equiv 1$ for every $s\geqq 0$. In that case $\chi\Big(h(s)\Big)\equiv\chi(1)=0$ such that

$$\lim_{n\to+\infty} M^n\Big(1-\varphi(\delta_n s)\Big)=0,$$

which proves the necessity of our first condition. In the other case there exists a number $s_0>0$ such that $h(s_0)<1$.

As we see, $\chi\Big(h(s)\Big)$ is not positive and for increasing s it decreases. For $s=s_0$ we have thus

$$\lim_{n\to+\infty}\Big(\varphi(\delta_n s_0)-1\Big)\,M^n=\chi\Big(h(s_0)\Big)<0,$$

because $h(s_0)<1$.

Confering the two last limit relations we see that

$$\lim_{n\to+\infty} \frac{1-\varphi(\delta_n s)}{1-\varphi(\delta_n s_0)} = \lim_{n\to+\infty} \frac{1-\varphi\left(\delta_n s_0 \frac{s}{s_0}\right)}{1-\varphi(\delta_n s_0)} = \frac{\chi\left(h(s)\right)}{\chi\left(h(s_0)\right)},$$

exists. On the other hand, the conditions Lemma 1. of Feller's book [8], page 335, are fulfilled for the function

$$U(s) = 1-\varphi(s).$$

So

$$\lim_{n\to+\infty} \frac{1-\varphi(\delta_n s)}{1-\varphi(\delta_n s_0)} = \left(\frac{s}{s_0}\right)^{\alpha}. \qquad (-\infty<\alpha<+\infty) \tag{17}$$

These limit relations show that

$$\chi\left(h(s)\right) = \chi\left(h(s_0)\right) s^{\alpha} = C^* s^{\alpha}, \tag{18}$$

where $C^* = \chi\left(h(s_0)\right) s_0^{\alpha} < 0$. (18) implies that

$$h(s) = \chi^{-1}(C^* s^{\alpha}). \tag{19}$$

Now $h(s)$ as the Laplace transform of the limiting distribution function is monotonically decreasing and convex for $s>0$. So, by (18) the case $\alpha<0$ is impossible.

By (19) the derivative of $h(s)$ for $s>0$ is the following:

$$h'(s) = \frac{C^* \alpha s^{\alpha-1}}{\chi'\left(h(s)\right)}$$

Now if $\alpha>1$ then $h'(0)=0$, which is not possible because in this case we would have $h(s)\equiv 1$ $(s>0)$. This contradicts to the fact that $h(s_0)<1$. So, $0\leqq\alpha\leqq 1$. Now the case $\alpha=0$ is also impossible. In fact, if $\alpha=0$, then by (18)

$$h(s) = \chi^{-1}(C^*). \qquad (s\geqq 0)$$

But $C^*<0$ and $\chi^{-1}(z)$ is strictly monotonic. So $h(s)=\chi^{-1}(C^*)<1$. This contradicts to the fact that $h(s)$ is a Laplace transform.

Denoting $-C^*$ by C we obtain that

$$\lim_{n\to+\infty} M^n\left(1-\varphi(\delta_n s)\right) = Cs^{\alpha}, \tag{20}$$

where $C>0$ is a constant and $0<\alpha\leqq 1$.

Sufficiency. If (20) is satisfied then

$$\varphi(\delta_n s) = 1 - \frac{Cs^{\alpha}}{M^n}\left(1+o(1)\right).$$

Now for fixed $s\geqq 0$

$$\left| 1 - \frac{Cs^{\alpha}}{M^n}\left(1+o(1)\right) - e^{-\frac{Cs^{\alpha}}{M^n}\left(1+o(1)\right)} \right| = o\left(\frac{1}{M^n}\right).$$

This means that

$$\varphi(\delta_n s) = e^{-\frac{Cs^{\alpha}}{M^n}\left(1+o(1)\right)} + o\left(\frac{1}{M^n}\right),$$

and so the Laplace transform of (14) is

$$f_n\left(\varphi(\delta_n s)\right)=f_n\left(e^{-\frac{Cs^\alpha}{M^n}\left(1+o(1)\right)}+o\left(\frac{1}{M^n}\right)\right).$$

Now by Lagrange's theorem

$$\left|f_n\left(e^{-\frac{Cs^\alpha}{M^n}\left(1+o(1)\right)}+o\left(\frac{1}{M^n}\right)\right)-f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right)\right|=$$

$$=\left|f_n'\left(\vartheta(s)\right)\left[e^{-\frac{Cs^\alpha}{M^n}}\left(e^{-\frac{Cs^\alpha}{M^n}v(1)}-1\right)+o\left(\frac{1}{M^n}\right)\right]\right|\leqq$$

$$\leqq M^n\left(\frac{o(1)}{M^n}+o\left(\frac{1}{M^n}\right)\right)=o(1).$$

So, if the limit

$$\lim_{n\to+\infty} f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right) \tag{21}$$

exists, then there exists also the limit of the Laplace transform of (14). To prove the existence of (21) let us consider the Laplace transform of the random variable $Z_i^{(n)}/M^n$, where $Z_i^{(n)}$ is defined by (8). The value of this transform at the point $Cs^\alpha\,(s\geqq 0)$ is exactly

$$f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right)$$

By our suppositions the Lemma of Section 2. applies, and so

$$\lim_{n\to+\infty} f_n\left(e^{-\frac{Cs^\alpha}{M^n}}\right)=h(s)$$

exists and it is the Laplace transform of a distribution function. The argumentation is similar in the case when $\varphi(\delta_n s)=s+o\left(\frac{1}{M^n}\right)$. We get in this case $h(s)\equiv 1$. This proves our theorem.

Gnedenko and Freier in paper [5] proved that (20) is true if and only if the corresponding distribution function

$$F(x)=P(\tau_i-\tau_{i-1}<x) \qquad (i=1,\ 2,\ \ldots)$$

is of the following from: for $0<\alpha<1$

$$\lim_{x\to+\infty}\frac{1-F(kx)}{1-F(x)}=k^{-\alpha} \tag{22}$$

for every $k>0$, and for $\alpha=1$

$$\lim_{x\to+\infty}\frac{x\left(1-F(x)\right)}{\int\limits_0^x\left(1-F(z)\right)dz}=0. \tag{23}$$

This enables us to give an other formulation of Theorem 4.

Theorem 5. *Under the hypotheses of Theorem 4. the Laplace transform of the possible limiting distributions is one of the following expressions:*

$$h(s) \equiv 1 \qquad (s \geqq 0)$$

or

$$h(s) = \chi^{-1}(Cs^{\alpha}), \qquad (C > 0)$$

where $\chi^{-1}(z)$ *is the inverse of the solution of the Schröder's equation*

$$\chi\left(f(z)\right) = \frac{1}{M}\,\chi(z),$$

and $0 < \alpha \leqq 1$.

In order that the Laplace transform of the limiting distribution be

$$h(s) = \chi^{-1}(Cs) \qquad (C > 0)$$

it is necessary and sufficient that the distribution function of the original renewal process satisfy the relation

$$\lim_{x \to +\infty} \frac{x\left(1 - F(x)\right)}{\int\limits_0^x \left(1 - F(z)\right) dz} = 0.$$

In this case the limiting distribution belongs to the class of those distributions which can be obtained as limiting distributions for Galton — Watson processes with $f'(1) = M > 1$. *Let now* $0 < \alpha < 1$. *In order the Laplace transform of the limiting distribution be*

$$h(s) = \chi^{-1}(Cs^{\alpha}) \qquad (C > 0),$$

it is necessary and sufficient that for every fixed

$$\lim_{x \to +\infty} \frac{1 - F(kx)}{1 - F(x)} = k^{-\alpha}$$

be satisfied.

Finally, $h(s) \equiv 1$, *if and only if the Laplace transform* $\varphi(s)$ *of* $F(x)$ *is of the following form*

$$1 - \varphi(\delta_n s) = o\left(\frac{1}{M^n}\right).$$

Theorem 6. *If we prescribe the value of* δ_n *to be* $\frac{1}{M^n}$ *then the limiting distribution of* (14) *exists if and only if the mean value* μ *of the distribution function* $F(x)$ *is finite.*

Proof. In fact, if μ is finite, then Theorem 3. shows that

$$\frac{1}{M^n}\left(\tau_i^{(n)*} - \tau_{i-1}^{(n)*}\right) \tag{24}$$

has a limiting distribution $G(x)$. Conversely, if (2) has a limiting distribution, then by Theorem 4.

$$\lim_{n \to +\infty} M^n\left(1 - \varphi\left(\frac{s}{M^n}\right)\right) = 0,$$

or

$$\lim_{n \to +\infty} M^n\left(1 - \varphi\left(\frac{s}{M^n}\right)\right) = Cs^{\alpha}. \qquad (0 < \alpha \leqq 1,\ C > 0).$$

From the first relation we deduced that the Laplace transform of the limiting distribution is of the form

$$h(s) \equiv 1.$$

This case is irrelevant. If the second limit relation is true then

$$\lim_{n\to+\infty} \int_0^{+\infty} \frac{1-e^{-\frac{sx}{M^n}}}{\frac{s}{M^n}}\, dF(x) = Cs^{\alpha-1}.$$

By Fatou's lemma it follows from this that

$$\mu = \int_0^{+\infty} J dF(J) \leqq \lim_{n\to+\infty} \int_0^{+\infty} \frac{1-e^{-\frac{sx}{M^n}}}{\frac{s}{M^n}}\, dF(x) = Cs^{\alpha-1}, \qquad (s>0)$$

which proves our assertion.

4. Theorem 6. has an interesting consequence. In fact, we can raise the following problem: what are the renewal prosesses $\tau_0 \equiv 0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots$ which remain invariant under the rarefaction and compression procedure expressed by the formulas (8), (9) and (10). Under invariance with respect to this rerefaction and compression we mean that after the n-th step the distribution function of the new process $t_0^{(n)} \equiv 0 \leqq t_1^{(n)} \leqq \ldots$ $(n=1, 2, \ldots)$ is the same as that before the rarefaction and compression.

Theorem 7. *If* $0 < D^2(\nu_i^{(n)}) < +\infty$, $(i, n = 1, 2, \ldots)$, *the sole invariant renewal processes with respect to the rarefaction and compression procedure, expressed by* k8), (9) *and* (10), *are those whose distribution function is* $G\left(\frac{x}{\mu}\right)$, *where* $G(x)$ *is defined by the Lemma and* μ *is an arbitrary finite positive number.*

Proof. Let $h(s)$ denote the Laplace transform of $G(x)$. Then the Laplace transform of the distribution function of $t_i^{(1)} - t_{i-1}^{(1)}$ $(i=1, 2, \ldots)$ is $f\left(h\left(\frac{s}{M}\right)\right.$. It is known in the theory of the Galton—Watson processes that

$$f\left(h\left(\frac{s}{M}\right)\right) = h(s).$$

Since, for finite $\mu > 0$, $h(\mu s)$ also satisfies this equation, it follows that the renewal processes with distribution function $G\left(\frac{x}{\mu}\right)$ are invariant.

Conversely, let $F(x)$ be the distribution function of the invariant process $\tau_0 \equiv \equiv 0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots$, and let $\varphi(s)$ denote its Laplace transform. Then the invariance of the process implies that

$$\varphi(s) = f\left(\varphi\left(\frac{s}{M}\right)\right).$$

From this it follows that

$$\varphi(s) = f_n\left(\varphi\left(\frac{s}{M^n}\right)\right). \qquad (n = 1,\ 2,\ \ldots)$$

Let now $n \to +\infty$. Then

$$\lim_{n \to +\infty} f_n \left(\varphi \left(\frac{s}{M^n} \right) \right)$$

exists and equals $\varphi(s)$. From this by Theorem 6. we deduce that $F(x)$ has finite mean value μ. Theorem 3. now implies that $F(x) = G\left(\frac{x}{\mu}\right)$.

Theorem 7 has been proved partially by the autnor in [3] and by other methods by T. Szantai [4]. His proof is complete.

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НЕСКОЛЬКО ЗАМЕЧАНИЙ О РАЗРЕЖЕНИИ РЕКУРРЕНТНЫХ ПОТОКОВ

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(*Резюме*)

Имеется рекуррентный поток событий $\tau_0 \equiv 0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots$ с функцией распределения $F(x)$ длин промежутков времени между последовательными событиями. Пусть $\nu_i^{(n)}$ — последовательность независимых и одинаково распределенных случайных величин, принимающих положительные целые значения $(i, n = 1, 2, \ldots)$, не зависящая от потока. Сделаем следующую операцию: возьмем те события из потока, индексы которых $\nu_1^{(1)}, \nu_2^{(1)}, \ldots$ Получаем новый поток

$$\tau_0^{(1)} \equiv 0 \leqslant \tau_1^{(1)} = \tau_{\nu_1^{(1)}} \leqslant \tau_2^{(1)} = \tau_{\nu_1^{(1)} + \nu_2^{(1)}} \leqslant \ldots$$

Повторим указанную операцию n раз рекурсивно: во втором шагу операция делается над потоком $\tau_0^{(1)} \leqq \tau_1^{(1)} \leqslant \tau_2^{(1)} \leqslant, \ldots,$ но сейчас со случайными разреживающими величинами $\nu_1^{(2)}$ $\nu_2^{(2)}$, ,,,, а в n-том шагу над потоком $\tau_0^{(n-1)} \equiv 0 \leqq \tau_1^{(n-1)} \leqq \tau_2^{(n-1)} \leqslant \ldots$ и со случайными величинами $\nu_1^{(n)}, \nu_2^{(n)}, \ldots$.

Эта операция — обобщение операции A. Re'nyi так как у него $\nu_i^{(n)}$ распределены по закону $P(\nu_i^{(n)} = k) = q(1-q)^{k-1}$, $0 < q < 1$, $k = 1, 2, \ldots$

Мы предполагаем, что $0 < D^2(\nu_i^{(n)}) = \sigma^2 < +\infty$. Исследуется предельное распределение разниц

$$\delta_n(\tau_i^{(n)} - \tau_{i-1}^{(n)})$$

при фиксированном i, когда $\delta_n \to 0$ при $n \to +\infty$.

Необходимое и достаточное условие для существования предельного распределения дается в теореме 4 и в другой формулировке в теореме 5.

KELIOS PASTABOS APIE REKURENTINIŲ SRAUTŲ IŠRETINIMĄ

J. Moderudi

(*Reziumė*)

Darbe nagrinėjama atsitiktinių dydžių seka

$$\tau_0 = 0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \ldots,$$

kuri nusako rekurentinį įvykių srautą. Tarkime, kad $\nu_i^{(n)}$ – nepriklausomų vienodai pasiskirsčiusių dydžių, įgyjančių reikšmes nepriklausomai nuo srauto dydžių, seka. Pažymime

$$\tau_i^{(1)} = \tau_i, \quad \tau_i^{(n)} = \tau^{(n-1)}_{\nu_1^{(n)} + \ldots + \nu_i^{(n)}}. \qquad (n \geqslant 2,\ i = 0,\ 1,\ \ldots)$$

Esant sąlygai

$$0 < D^2(\nu_i^{(n)}) = \sigma^2 < +\infty,$$

4 teoremoje (kita formuluotė 5 teoremoje) gautos būtinos ir pakankamos skirtumų

$$\delta_n(\tau_i^{(n)} - \tau_{i-1}^{(n)}$$

konvergavimo į ribinį dėsnį, kai $\delta_n \to 0$, $n \to \infty$, sąlygos.

Gautas rezultatas apibendrina A. Renyi teoremą, įrodytą specialiam dydžių $\nu_i^{(n)}$ atvejui.