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Investigation of complex eigenvalues for a stationary problem with two-point nonlocal boundary condition

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Abstract. The Sturm–Liouville problem with one classical and another two-point nonlocal boundary condition is considered in this paper. These problems with nonlocal boundary condition are not self-adjoint, so the spectrum has complex points. We investigate how the spectrum in the complex plane of these problems (and for the Finite-Difference Schemes) depends on parameters γ and ξ of the nonlocal boundary conditions.

Keywords: complex eigenvalues, two-point Nonlocal Boundary Condition, Finite-Diference Scheme.

1 Introduction

Problems with Nonlocal Boundary Conditions (NBCs) are an area of the fast developing differential equations theory. Problems of this type arise in various fields of physics, biology, biotechnology. A. Samarskii and A. Bitsadze formulated and investigated a nonlocal boundary problem for an elliptic equation [1]. A nonlocal boundary problem for the second order ordinary differential equation was investigated in [2]. An eigenvalue problem with the nonlocal condition is closely linked with a boundary problem for a differential equation with NBC [3]. Complex eigenvalues were investigated in [4].

2 The Sturm–Liouville problem with NBC

Let us investigate the Sturm–Liouville Problem (SLP)

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1}$$

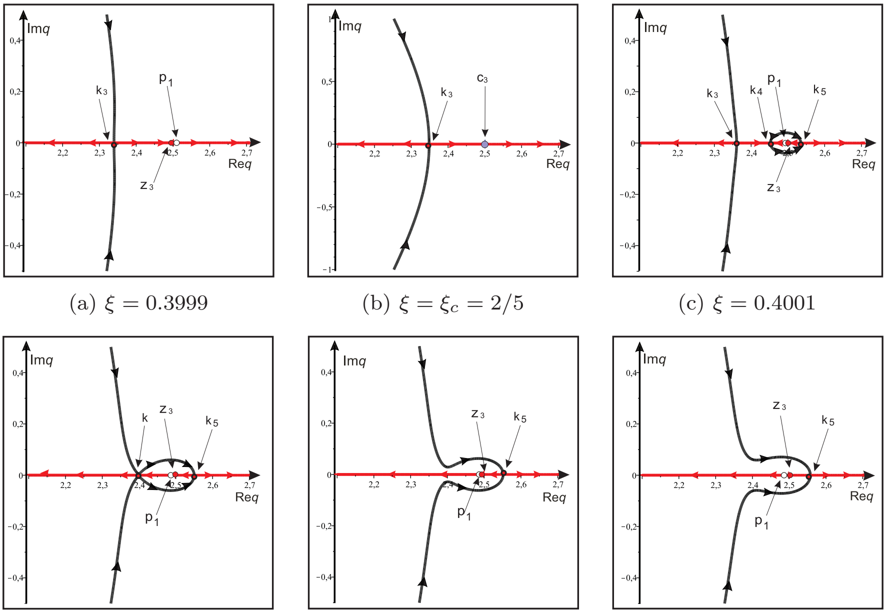
with one classical (the first or the second type) BC:

$$u(0) = 0 \quad \text{or} \quad u'(0) = 0, \tag{$2_{1,2}$}$$

and another two-point BC ($0 \leqslant \xi \leqslant 1$):

$$u(1) = \gamma u'(\xi) \quad \text{or} \quad u(1) = \gamma u(\xi), \tag{$3_{1,2}$}$$

with the parameter $\gamma \in \mathbb{R}$.



(a) $\xi = 0.3999$ (b) $\xi = \xi_c = 2/5$ (c) $\xi = 0.4001$

(d) $\xi = 0.400218...$ (e) $\xi = 0.400235$ (f) $\xi = 0.40036$

Fig. 1. Domain $\mathcal{N}$ with various ξ for the complex-real fubction $\gamma_1(\pi\xi)$.

2.1 Differential problem with the Neumann BC and NBC

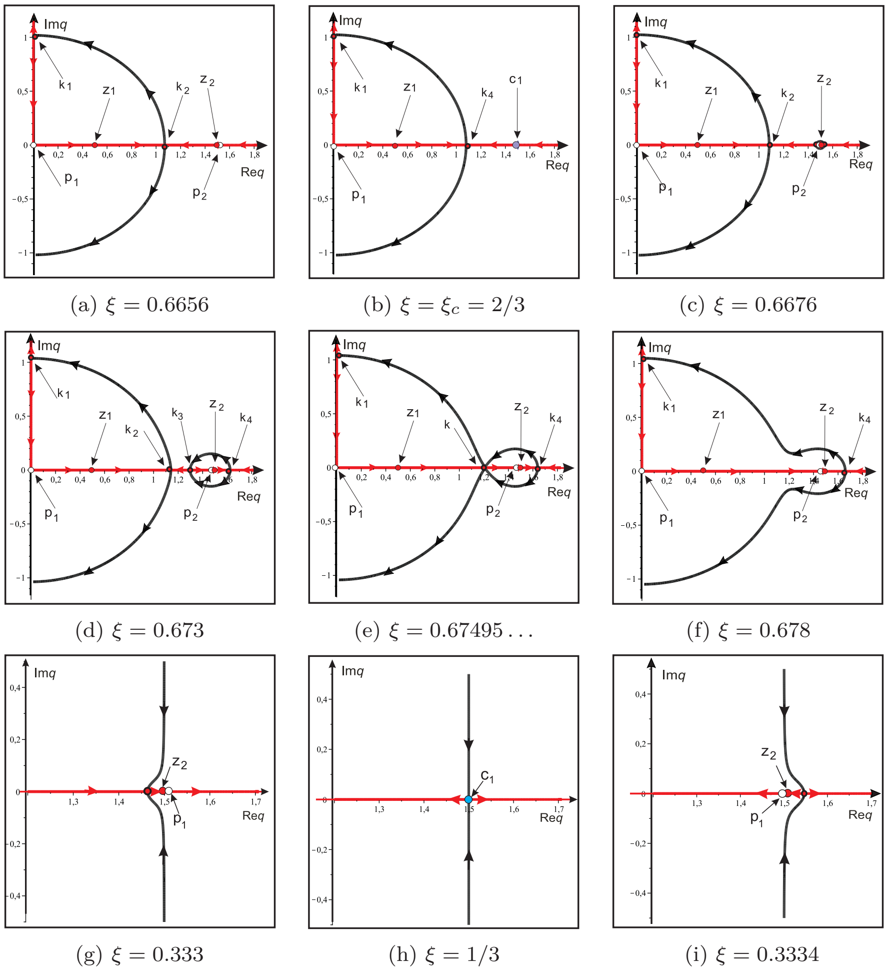
Let us investigate SLP (1) with one BC (2_2) and another two-point BC (3). As $\gamma = 0$ in problem (1),(2_2)–(3), we get a problem with classical BCs. Then eigenvalues and eigenfunctions don't depend on the parameter ξ: $\lambda_k = \pi^2(k-1/2)^2$, $u_k = \cos(\pi(k-1/2)t)$, $k \in \mathbb{N}$. For $\gamma \neq 0$, the eigenvalues $\lambda = q^2$, $q \in \mathbb{C}_q := \{q \in \mathbb{C}\colon \operatorname{Re} q > 0 \text{ or } \operatorname{Re} q = 0,\ \operatorname{Im} q > 0 \text{ or } q = 0\}$ and eigenfunctions are $u = \cos(qt)$.

Constant eigenvalues (which don't depend on the parameter γ, see [4]) do not exist for irrational ξ, while for rational $\xi = r = m/n \in [0,1]$ they exist in the following cases: $m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$ (Case 1); $m \in \mathbb{N}_o$, $n \in \mathbb{N}_o$, $m \leqslant n$ (Case 2). Constant eigenvalues are equal to $\lambda_k = (c_k)^2$, $c_k = \pi(k-1/2)n$, $k \in \mathbb{N}$, where c_k is a constant eigenvalue point, and $\mathbb{N}_e :=\{0,2,4,6,...\}$ and $\mathbb{N}_o :=\{1,3,5,...\}$.

We also have nonconstant eigenvalues which depend on the parameter γ and are γ-values of complex-real ($\mathbb{C}$-$\mathbb{R}$) characteristic functions

$$\gamma_1(q) = -\cos q / \bigl(q \sin(\xi q)\bigr), \qquad \gamma_2(q) = \cos q / \cos(\xi q). \tag{$4_{1,2}$}$$

The points $z_k = \pi(k-1/2)$, $k \in \mathbb{N}$, are zeroes (the first order) of the meromorphic function $\gamma(q)$. The points $p_k = k\pi/\xi$, $k \in \mathbb{N}$ (Case 1) and $p_k = \pi(k-1/2)/\xi$, $k \in \mathbb{N}$ are poles (the first order). The restriction function $\gamma(q)$ defines a subset (*net*) $\mathcal{N} := \gamma^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_q\colon \operatorname{Im} \gamma_c(q) = 0\} \subset \mathbb{C}_q$. In the general case, the subset $\mathcal{N}$ is a union of curves. Two different curves can intersect at critical points. We add arrows that show how the eigenvalue points are moving, i.e., the direction in which the parameter γ is growing (see [4]). In Case 1, domain $\mathcal{N}$ is presented in Figs. 1 and 2(a)–2(f). In Case 2, the spectrum is more simple than in Case 1 (see Figs. 2(g)–2(i)).



(a) $\xi = 0.6656$ (b) $\xi = \xi_c = 2/3$ (c) $\xi = 0.6676$

(d) $\xi = 0.673$ (e) $\xi = 0.67495\ldots$ (f) $\xi = 0.678$

(g) $\xi = 0.333$ (h) $\xi = 1/3$ (i) $\xi = 0.3334$

Fig. 2. Domain $\mathcal{N}$ with various ξ for the complex-real function $\gamma_1(\pi\xi)$ (a)–(f) and $\gamma_2(\pi\xi)$ (g)–(i).

2.2 The discrete Sturm–Liouville problem with NBC

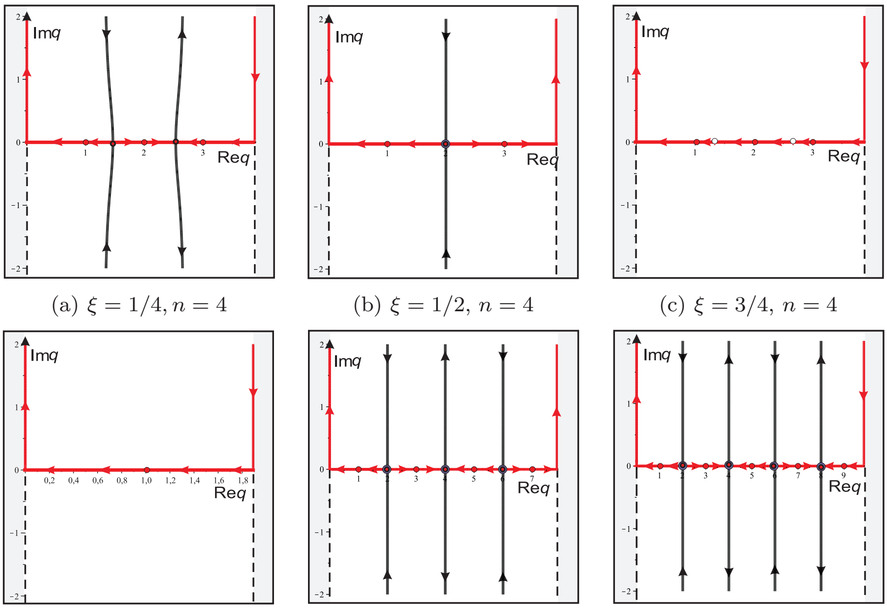
Let us investigate SLP (1) with one classical BC (2_1) and another two-point BC (3) with the parameter $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. In the interval $[0,1]$, we introduce a uniform grid $\overline{\omega}^h = \{t_j = jh,\ j = \overline{0,n};\ nh = 1\}$. We make an assumption that ξ is coincident with the grid point, i.e., $\xi = mh = m/n$, where $m = \overline{1, n-1}$. We approximate SLP (1),(2_1)–(3) by the following Finite-Difference Scheme (FDS):

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \qquad U_0 = 0, \tag{5}$$

$$U_n = \gamma U_m, \qquad U_n = \frac{\gamma}{2h}(U_{m+1} - U_{m-1}). \tag{$5_{1,2}$}$$

Let us denote the greatest common divisor $K := \gcd(n,m)$ and $N := n/K$, $M := m/K$. Then $\xi = M/N$, too. Rewrite Eq. (5) in another form:

$$U_{j+1} - 2\cos(\pi q h)U_j + U_{j-1} = 0, \qquad \lambda = \frac{4}{h^2}\sin^2\left(\frac{\pi q h}{2}\right), \qquad U_j = \sin(\pi q t_j), \tag{6}$$



(a) $\xi = 1/4, n = 4$ (b) $\xi = 1/2,\ n = 4$ (c) $\xi = 3/4,\ n = 4$

(d) $\xi = 1/2, n = 2$ (e) $\xi = 1/2,\ n = 8$ (f) $\xi = 1/2,\ n = 10$

Fig. 3. Domain $\mathcal{N}$ for a discrete problem in Case 1.

where $q = x + iy \in \mathbb{C}_q^h = \{q\colon 0 < x < n\} \cup \{q\colon x = 0, y \geqslant 0\} \cup \{q\colon x = n, y \geqslant 0\}$.

Constant eigenvalue points are equal to $c_k := Nk$, $k = \overline{1, K-1}$ (Case 1), and $c_k := N(k - 1/2)$ (Case 2), $k = \overline{1, K}$. Nonconstant eigenvalues, as γ-values of the $\mathbb{C}$-$\mathbb{R}$ characteristic function, are defined on the set $\mathbb{C}_q^h$:

$$\gamma_1 = \frac{\sin(\pi q)}{\sin(\pi q \xi)}, \qquad \gamma_2 = \frac{\sin(\pi q)}{q \cos(\pi q \xi)} \cdot \frac{qh}{\sin(\pi q h)}. \tag{$7_{1,2}$}$$

Domain $\mathcal{N}$ is shown in Fig. 3 (Case 1) and in Fig. 4 (Case 2). We can see, how the spectrum depends on the number of the grid points n, where the ξ value is the same.

2.3 The discrete Sturm–Liouville problem with a Neumann BC and NBC

We approximate differential problem (1)(2_2)–(3) by the following FDS:

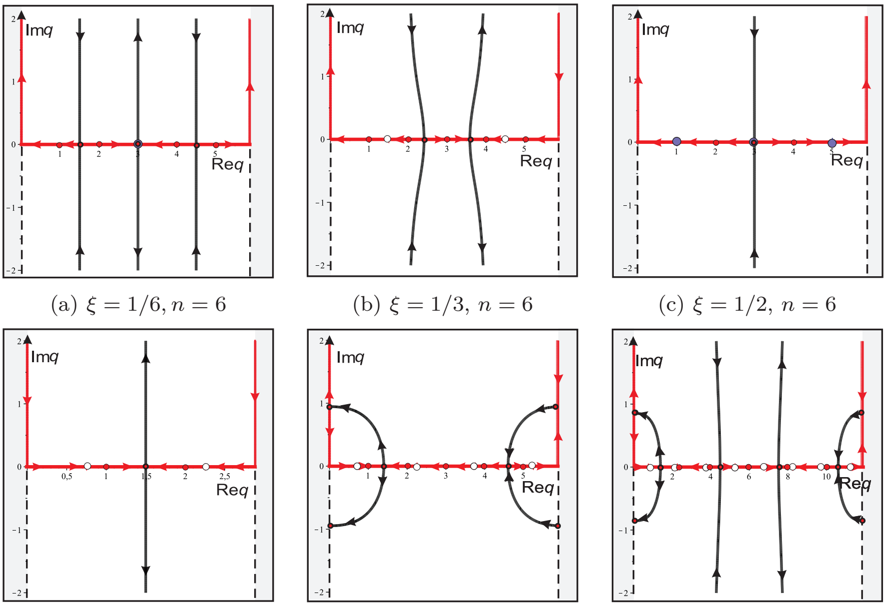
$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \qquad U_1 = U_0, \tag{8}$$

$$U_n = \gamma U_m, \qquad U_n = \frac{\gamma}{2h}(U_{m+1} - U_{m-1}). \tag{$8_{1,2}$}$$

Let us denote the greatest common divisor $K := \gcd(2n - 1, 2m - 1)$, $N := n/K$, $L := \lfloor K/2 \rfloor$. Then $\xi = M/N$, too. Rewrite Eq. (8) in another form (6) as for FDS (5):

$$U_{j+1} - 2\cos(\pi q h) U_j + U_{j-1} = 0, \qquad \lambda = \frac{4}{h^2} \sin^2\left(\frac{\pi q h}{2}\right),$$

$$U_j = \cos\left(\pi q \left(t_j - \frac{h}{2}\right)\right). \tag{9}$$



(a) $\xi = 1/6, n = 6$ (b) $\xi = 1/3$, $n = 6$ (c) $\xi = 1/2$, $n = 6$

(d) $\xi = 2/3, n = 3$ (e) $\xi = 2/3$, $n = 6$ (f) $\xi = 2/3$, $n = 12$

Fig. 4. Domain $\mathcal{N}$ for a discrete problem in Case 2.

If $\gamma = 0$, we have classical BCs and all $n-1$ eigenvalues for classical FDS are positive and algebraically simple and do not depend on the parameter ξ:

$$\lambda_k^0 = \frac{4}{h^2}\sin^2\left(\pi q_k^0 h/2\right), \qquad U_j^{k;0} = \cos\left(\pi q_k^0\left(t_j - \frac{h}{2}\right)\right),$$
$$q_k^0 = \frac{2k-1}{2-h}, \quad k = \overline{1, n-1}. \tag{10}$$

We have the eigenvalue $\lambda = 0$ for problem (8)–($8_{1,2}$) if and only if $\gamma = 1$ in Case 1, while in Case 2 the eigenvalue $\lambda = 0$ does not exist.

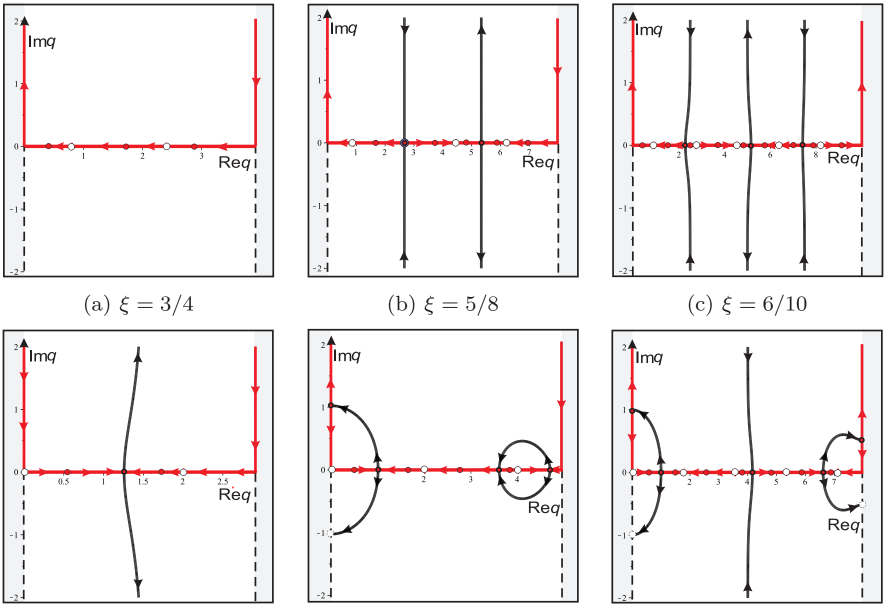
In Case 1, constant eigenvalue points are equal to $c_k := Nk$, $k = \overline{1, L}$, and in Case 2, constant eigenvalues do not exist. We can find nonconstant eigenvalues with the help of the $\mathbb{C}$–$\mathbb{R}$ characteristic function:

$$\gamma_1 = \frac{\cos(\pi q((2-h)/2))}{\cos(\pi q((2\xi - h)/2))}, \qquad \gamma_2 = -\frac{\cos(\pi q((2-h)/2))}{q\sin(\pi q((2\xi - h)/2))}\cdot\frac{qh}{\sin(\pi q h)}. \tag{$11_{1,2}$}$$

The Domain $\mathcal{N}$ for discrete problem in Cases 1 and 2 is shown in Fig. 5.

3 Conclusion

The spectrum of differential problem (1),(2_2)–(3) and FDS (11)–($11_{1,2}$) is different: in Case 1 of FDS, there exist only real eigenvalues if $m = n-1$; in Case 2 of FDS, constant eigenvalues do not exist, there exists a similar ring of complex eigenvalues near to the axis of the γ function value as in the differential problem, but if $m = n-1$, the such ring does not exist. The spectrum of FDS problem (7)–($7_{1,2}$) is very similar to that of problem (11)–($11_{1,2}$): in Case 1, there exist only real eigenvalues if $m = n-1$, and in Case 2, constant eigenvalues do not exist, but there are rings if $m < n-1$.



(a) $\xi = 3/4$ (b) $\xi = 5/8$ (c) $\xi = 6/10$

(d) $\xi = 2/3$ (e) $\xi = 3/5$ (f) $\xi = 5/8$

Fig. 5. Domain $\mathcal{N}$ for a discrete problem: Case 1 (a)–(c) and Case 2 (d)–(f).

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REZIUMĖ

Stacionariojo uždavinio su viena dvitaške nelokaliąja sąlyga kompleksinių tikrinių reikšmių tyrimas

K. Skučaitė-Bingelė, A. Štikonas

Šiame straipsnyje yra nagrinėjamas Šturmo ir Liuvilio uždavinys su viena klasikine ir antra nelokalia dvitaške kraštine sąlyga. Šie uždaviniai su nelokaliomis kraštinėmis sąlygomis nėra sau jungtiniai, todėl tikrinės reikšmės gali būti kompleksinės. Ištirta diferencialinio uždavinio ir baigtinių skirtumų schemos kompleksinės spektro dalies priklausomybė nuo nelokaliųjų kraštinių sąlygų parametrų γ ir ξ. Dauguma tyrimo rezultatų pateikiama $\mathbb{C}$-$\mathbb{R}$ charakteristinės funkcijos grafikais.

Raktiniai žodžiai: kompleksinės tikrinės reikšmės, dvitaškės nelokaliosios kraštinės sąlygos, baigtinių skirtumų schema.