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The properties of Green's functions for one stationary problem with nonlocal boundary conditions

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Abstract. In this paper we research Green's function properties for stationary problem with four-point nonlocal boundary conditions. Dependence of these functions on values ξ and γ is investigated. Green's functions graphs with various values ξ and γ are presented.

Keywords: stationary differential problem, Green's function, nonlocal boundary conditions.

1. Introduction

In this paper we consider inhomogeneous two-order differential equation with two nonlocal boundary conditions

$$-u'' = f, \quad x \in (0,1), \tag{1}$$

$$u(0) = \gamma_0 u(\xi_0), \tag{2}$$

$$u(1) = \gamma_1 u(\xi_1), \tag{3}$$

where $f \in C[0,1]$, $\xi_0, \xi_1 \in [0,1]$, $\gamma_0, \gamma_1 \in \mathbb{R}$. This problem becomes classical for $\gamma_0 = \gamma_1 = 0$. Similar problems have been investigated in articles [1–4,6].

In [1] problems with more general nonlinear right-hand side $f(x,u,u')$ of differential equation were investigated. Also sufficient conditions for existence of positive solutions were found. In [6] Green's functions for linear case were considered, when one boundary condition is classical and another condition is two-point nonlocal boundary condition. The second-order differential equation with two additional conditions was investigated in [5]. In this paper expression of Green's function was derived. Some examples of various boundary conditions were presented in [2,3].

Green's function of this problem exists and is unique if $\theta = \theta(\gamma_0, \gamma_1, \xi_0, \xi_1) := 1 - \gamma_0(1-\xi_0) - \gamma_1\xi_1 - \gamma_0\gamma_1(\xi_0 - \xi_1) \neq 0$. Then Green's function has the following form (see [3,5]):

$$\begin{aligned} G(x,s) = G^{\mathrm{cl}}(x,s) &- \frac{\gamma_0(x - 1 - \gamma_1(x-\xi_1))}{1 - \gamma_0(1-\xi_0) - \gamma_1\xi_1 - \gamma_0\gamma_1(\xi_0-\xi_1)} \cdot G^{\mathrm{cl}}(\xi_0, s) \\ &+ \frac{\gamma_1(x - \gamma_0(x-\xi_0))}{1 - \gamma_0(1-\xi_0) - \gamma_1\xi_1 - \gamma_0\gamma_1(\xi_0-\xi_1)} \cdot G^{\mathrm{cl}}(\xi_1, s), \end{aligned} \tag{4}$$

where G^{cl} is classical Green's function of problem (1)–(3) when $\gamma_0 = \gamma_1 = 0$ is

$$G^{\mathrm{cl}}(x,s) = \begin{cases} s(1-x), & s \leqslant x, \\ x(1-s), & x \leqslant s. \end{cases} \tag{5}$$

Let's denote

$$g_0 = g_0(x,\gamma_0,\gamma_1,\xi_1) := -\gamma_0\big(x - 1 - \gamma_1(x-\xi_1)\big),$$
$$g_1 = g_1(x,\gamma_0,\gamma_1,\xi_0) := \gamma_1\big(x - \gamma_0(x-\xi_0)\big).$$

Therefore Green's function can be written down in a following form:

$$G(x,s) = G^{\mathrm{cl}}(x,s) + \theta^{-1}(\gamma_0,\gamma_1,\xi_0,\xi_1)\big(g_0(x,\gamma_0,\gamma_1,\xi_1)\cdot G^{\mathrm{cl}}(\xi_0,s) + g_1(x,\gamma_0,\gamma_1,\xi_0)\cdot G^{\mathrm{cl}}(\xi_1,s)\big). \tag{6}$$

In this paper we investigate properties of Green's functions in relation to parameters γ_0, γ_1, ξ_0, ξ_1.

2. Green's function properties in the case $\xi_0 = \xi_1$

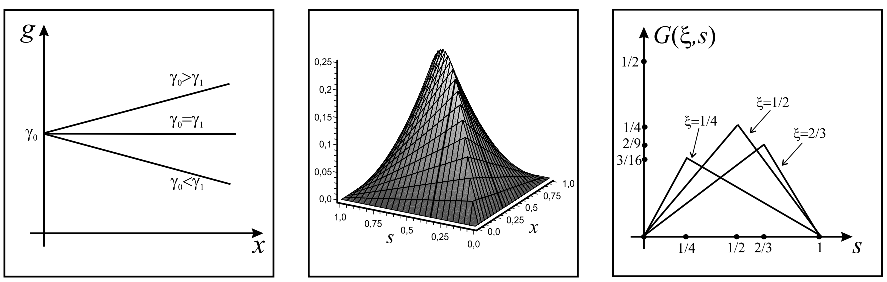
Let $\xi = \xi_0 = \xi_1$. Then formula (6) for Green's function is

$$G(x,s) = G^{\mathrm{cl}}(x,s) + \theta^{-1} g \cdot G^{\mathrm{cl}}(\xi,s), \tag{7}$$

where $g = g(x,\gamma_0,\gamma_1) := \gamma_0 + x(\gamma_1-\gamma_0)$ and $\theta(\gamma_0,\gamma_1,\xi) := 1 - \xi(\gamma_1-\gamma_0) - \gamma_0$.

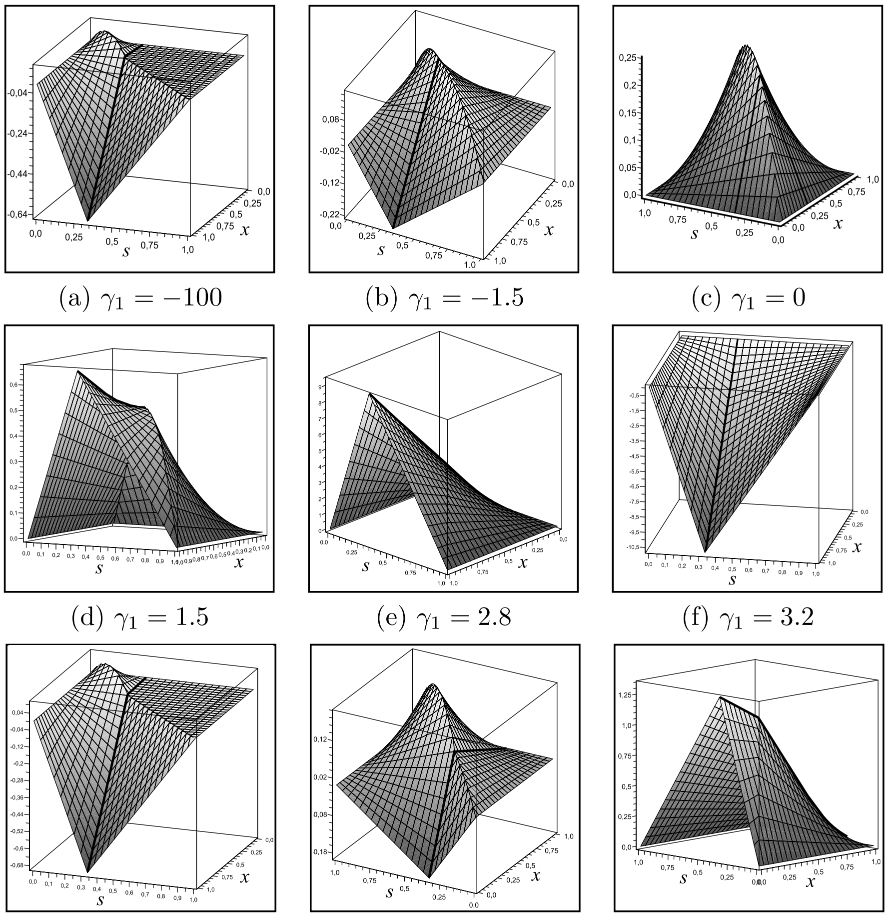
Dependence of function g from values γ_i, $i = 0,1$, is represented on Fig. 1(a). If $\gamma_0 > \gamma_1$, then function g increases; if $\gamma_0 = \gamma_1$, then function g is constant; and if $\gamma_0 < \gamma_1$, then function g decreases. Fig. 1(b) is classical Green's function $G^{\mathrm{cl}}(x,s)$ (see, formula (5)). Dark line on this graph is line, when $s = \xi$. The graphs of function $G^{\mathrm{cl}}(\xi,s)$ for fixed ξ we can see in Fig. 1(c).

Example 1. Case $\xi_0 = \xi_1 = 1/3$. In this case existence condition of Green's function is $\theta = 1 - \frac{2}{3}\gamma_0 - \frac{1}{3}\gamma_1 \neq 0$. On Figs. 2(a)–(g) are shown Green's functions with



(a) (b) (c)

Fig. 1. (a) Function $g = \gamma_0 + x(\gamma_1 - \gamma_0)$; (b) classical Green's function; (c) function $G^{\mathrm{cl}}(\xi,s)$.



(a) $\gamma_1 = -100$ (b) $\gamma_1 = -1.5$ (c) $\gamma_1 = 0$

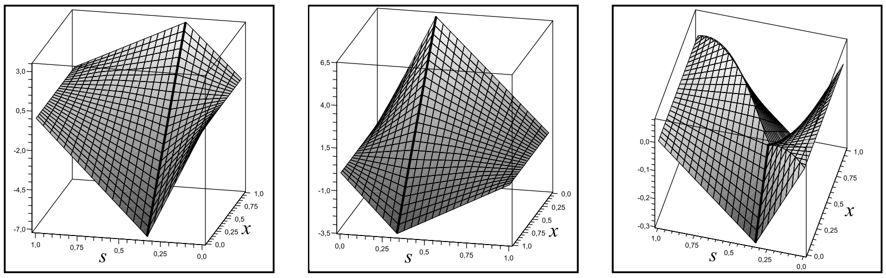
(d) $\gamma_1 = 1.5$ (e) $\gamma_1 = 2.8$ (f) $\gamma_1 = 3.2$

(g) $\gamma_1 = 100$ (h) $\gamma_0 = -2$ (i) $\gamma_0 = 1.2$

Fig. 2. Graphs of Green's function with $\xi_0 = \xi_1 = 1/3$ and (a)–(g) $\gamma_0 = 0$ and various $\gamma_1 = -100, -1.5, 0, 1.5, 2.8, 3.2, 100$; (h)–(i) $\gamma_1 = 0$ and various $\gamma_0 = -2, 1.2$.

$\gamma_0 = 0$ and various γ_1. When γ_1 converges to a minus or plus infinity (see Figs. 2(a) $\gamma_1 = -100$ and 2(g) $\gamma_1 = 100$), Green's functions are identical. Green's function, when $\gamma_1 \to 0$ (see Figs. 2(b), (d)) smoothly passed in classical Green's function (see Fig. 2(c)). For this problem, when $\gamma_0 = 0$, existence condition is $\gamma_1 \neq 3$. On Figs. 2(e) ($\gamma_1 = 2.8$) and (f) ($\gamma_1 = 3.2$) are shown functions where γ_1 converges to value, when Green's function does not exist, in our case to 3.

On Figs. 2(h)–(i) are shown Green's functions with various γ_0 and $\gamma_1 = 0$. If to replace in equation (4) $s \to 1 - s$, $x \to 1 - x$, $\xi_1 \to 1 - \xi_0$ and $\gamma_1 \to \gamma_0$, then again we will receive Green's function (4), as $G^{\mathrm{cl}}(1 - x, 1 - s) = G^{\mathrm{cl}}(x, s)$. Then a case, when $\gamma_1 = 0$ is similar to a case when $\gamma_0 = 0$.



(a) $\gamma_0 = 2.1,\ \gamma_1 = -1$ (b) $\gamma_0 = 1.9,\ \gamma_1 = -1$ (c) $\gamma_0 = 4,\ \gamma_1 = 4$

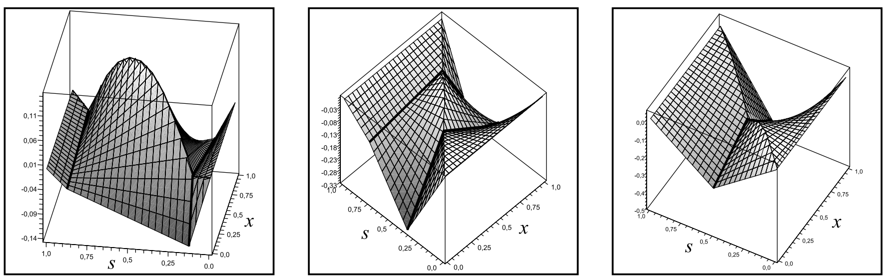
Fig. 3. Graphs of Green's function with $\xi_0 = \xi_1 = 1/3$.

On Fig. 3 are shown Green's functions with nonzero different parameters γ_0 and γ_1. Values γ_0 and γ_1 of Fig. 3(a), (b) converge to solutions of line $1 - \frac{2}{3}\gamma_0 - \frac{1}{3}\gamma_1 = 0$, when Green's function does not exist, in this case to $\gamma_0 = 2$, $\gamma_1 = -1$.

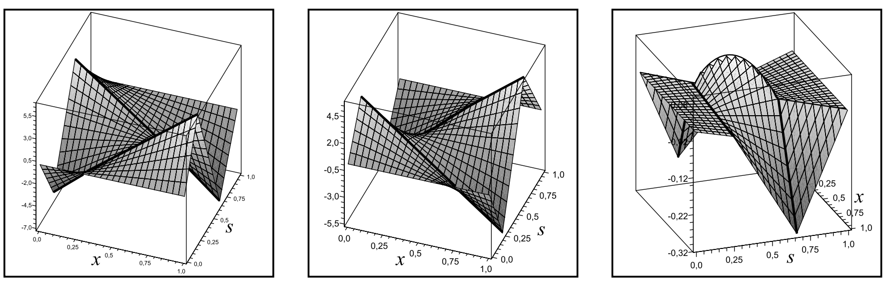
3. Green's function properties in the case $\xi_0 \neq \xi_1$

Now we consider the case, when ξ_0 and ξ_1 are different and investigated the formula (4) for Green's function.

In Fig. 4 we have graphs of Green's functions with different ξ_0 and ξ_1. On the Figs. 4(a)–(c) are fixed $\gamma_0 = 2$, $\gamma_1 = 4$. Green's function the discontinuity lines of derivative are $s = \xi_i$, $i = 0, 1$, and $s = x$.



(a) $\xi_0 = 7/8,\ \xi_1 = 1/8$ (b) $\xi_0 = 6/8,\ \xi_1 = 3/8$ (c) $\xi_0 = 1/2,\ \xi_1 = 1/2$



(d) $\gamma_0 = -2,\ \gamma_1 = -3$ (e) $\gamma_0 = -2,\ \gamma_1 = -2.7$ (f) $\gamma_0 = 15,\ \gamma_1 = 10$

Fig. 4. Graphs of Green's function with (a)–(c) $\gamma_0 = 2$, $\gamma_1 = 4$ and (d)–(f) $\xi_0 = 2/3$, $\xi_1 = 1/4$.

Example 2. Case $\xi_0 = \frac{2}{3}$, $\xi_1 = \frac{1}{4}$. On Figs. 4 (d)–(f) are fixed $\xi_0 = \frac{2}{3}$, $\xi_1 = \frac{1}{4}$. With these parameters existence condition is $12 - 4\gamma_0 - 3\gamma_1 - 5\gamma_0\gamma_1 \neq 0$. One of many non-existence points is $\gamma_0 = -2$, $\gamma_1 = -\frac{20}{7}$. In Figs. 4(d), (e) γ_0 and γ_1 converge to such points. Fig. 4(f) is graph of Green's function, when $\gamma_0 = 15$, $\gamma_1 = 10$.

4. Conclusions

The Green's functions properties for problem (1) with nonlocal boundary conditions (2) are similar to properties of the classical Green's function (i.e., Green's function for problem (1) with classical boundary conditions $\gamma_0 = \gamma_1 = 0$). Green's function existence conditions is $\theta = 1 - \gamma_0(1 - \xi_0) - \gamma_1\xi_1 - \gamma_0\gamma_1(\xi_0 - \xi_1) \neq 0$. For fixed ξ_0 and ξ_1 "bad" points lie on hyperbola or two lines or one line. Additionally, on lines $s = \xi_0$ and $s = \xi_1$ have discontinuities of derivative of the Green's function as in line $x = s$. Nonclassical Green's functions are nonsymmetrical.

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REZIUMĖ

S. Roman, A. Štikonas. Vieno stacionariojo uždavinio su nelokaliomis sąlygomis Gryno funkcijų savybės

Šiame straipsnyje mes nagrinėjame Gryno funkcijos savybes stacionariojo uždavinio su 4-taškėmis nelokaliosiomis sąlygomis. Tyriama šių funkcijų priklausomybė parametrų ξ ir γ atžvilgiu. Pateikti Gryno funkcijų grafikai su įvairiais parametrais.

Raktiniai žodžiai: stacionarus diferencialinis uždavinys, Gryno funkcija, nelokaliosios kraštinės sąlygos.