Lietuvos matematikos rinkinys. LMD darbai
ISSN 0132-2818
51 tomas, 2010, 379–384
www.mii.lt/LMR/

Linear ODE with nonlocal boundary conditions and Green's functions for such problems

Svetlana Roman, Artūras Štikonas

Institute of Mathematics and Informatics, Vilnius University
Akademijos 4, LT-08663 Vilnius
E-mail: svetlana.roman@ktl.mii.lt; ash@ktl.mii.lt

Abstract. In this article we investigate a formula for the Green's function for the n-order linear differential equation with n additional conditions. We use this formula for calculating the Green's function for problems with nonlocal boundary conditions.
Keywords: Green's function, nonlocal boundary conditions.

1 Intoduction

Let us investigate n-order differential equation with n additional conditions

$$\mathcal{L}u := u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1u' + a_0u = f, \quad x \in (0,l), \tag{1}$$
$$\langle L_i, u\rangle = 0, \quad i = 1,\dots,n, \tag{2}$$

where $L_1,\dots,L_n$ are linearly independent functionals in $C^n[0,l]$. We use notation $\boldsymbol{L} = (L_1,\dots,L_n)$.

The solution of this problem can be expressed via Green's function:

$$u(x) = \int_0^l G(x,y)f(y)\,dy.$$

where $G(x,y)$ is Green's function.

Green's function for problem (1)–(2) has been obtained in [8, 11] if $n = 2$ and in [9] if $n = 3$. Ma and Thompson [7] study positive solutions of second-order three-point boundary value problem and their existence condition. Such problem is investigated by Infante [4]. He uses the fixed point theory to establish the existence of positive solutions. Second-order m-point nonlocal problems also have been investigated by Lv and Pei in [6] and Rynne in [10]. Gou et al. [2], Liu and O'Regan [5], Henderson and Ntouyas [3] studied third-order and higher-order differential equations with various types of boundary conditions (BCs) and existence of solutions.

2 Formulae for Green's functions

Let $\boldsymbol{u} = [u_1,\dots,u_n]$ be fundamental system of homogeneous equation (1), and we denote

$$H_\theta(x,s) := \theta(x-s)\frac{\widetilde{W}[\boldsymbol{u}](s,x)}{W[\boldsymbol{u}](s)}, \qquad \theta(x) := \begin{cases} 1, & x > 0,\\ 0, & x \leqslant 0, \end{cases}, \quad x,s \in [0,l],$$

if $W[\boldsymbol{u}](s) \neq 0$, where

$$W[\boldsymbol{u}](s) := \begin{vmatrix} u_1(s) & \dots & u_n(s) \\ \dots & \dots & \dots \\ u_1^{(n-2)}(s) & \dots & u_n^{(n-2)}(s) \\ u_1^{(n-1)}(s) & \dots & u_n^{(n-1)}(s) \end{vmatrix}, \qquad \tilde{W}[\boldsymbol{u}](s,x) := \begin{vmatrix} u_1(s) & \dots & u_n(s) \\ \dots & \dots & \dots \\ u_1^{(n-2)}(s) & \dots & u_n^{(n-2)}(s) \\ u_1(x) & \dots & u_n(x) \end{vmatrix}$$

(see [1]). We use functional $\langle \delta_x, u\rangle = u(x)$ and determinants:

$$D(\boldsymbol{L})[\boldsymbol{u}] := \begin{vmatrix} \langle L_1, u_1\rangle & \dots & \langle L_n, u_1\rangle \\ \dots & \dots & \dots \\ \langle L_1, u_n\rangle & \dots & \langle L_n, u_n\rangle \end{vmatrix},$$

$$D(\boldsymbol{L}, \delta_x)[\boldsymbol{u}, G(\cdot,s)] := \begin{vmatrix} \langle L_1, u_1\rangle & \dots & \langle L_n, u_1\rangle & u_1(x) \\ \dots & \dots & \dots & \dots \\ \langle L_1, u_n\rangle & \dots & \langle L_n, u_n\rangle & u_n(x) \\ \langle L_1(\cdot), G(\cdot,s)\rangle & \dots & \langle L_n(\cdot), G(\cdot,s)\rangle & G(x,s) \end{vmatrix}.$$

The following theorems about Green's functions are valid.

Theorem 1. *Green's function for problem* (1), (2) *exists if* $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$ *and in this case it is equal to*:

$$G(x,s) = \frac{D(\boldsymbol{L}, \delta_x)[\boldsymbol{u}, H_\theta(\cdot,s)]}{D(\boldsymbol{L})[\boldsymbol{u}]}. \tag{3}$$

Theorem 2. *The relations between Green's functions* $G_v(x,s)$ *and* $G_u(x,s)$ *for problems*

$$\begin{cases} \mathcal{L}u = f, \\ \langle l_i, u\rangle = 0, \quad i = 1,\dots,n, \end{cases} \qquad \begin{cases} \mathcal{L}v = f, \\ \langle L_i, v\rangle = 0, \quad i = 1,\dots,n, \end{cases} \tag{4}$$

are:

$$G_v(x,s) = \frac{D(\boldsymbol{L}, \delta_x)[\boldsymbol{u}, G_u(\cdot,s)]}{D(\boldsymbol{L})[\boldsymbol{u}]}. \tag{5}$$

Proofs of these theorems are done in the case $n = 2$ [11], in the case $n = 3$ [9]. In general case the proof is similar.

3 Green's functions for problems with NBCs

In this section we investigate some examples with Nonlocal Boundary Conditions (NBCs), and construct Green's functions.

Example 1. Let us consider third order problem

$$u''' - \frac{1}{x+1}u'' - u' + \frac{1}{x+1}u = f, \quad x \in (0,1), \tag{6}$$

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(1) = \gamma u(\xi), \quad \xi \in (0,1). \tag{7}$$

We can take the fundamental system $u_1(x) = x+1$, $u_2(x) = e^x$, $u_3(x) = e^{-x}$ of homogeneous equation (6).

First we find Green's function for differential equation (6) with initial conditions (IC) $u(0) = u'(0) = u''(0) = 0$:

$$W[\boldsymbol{u}](s) = \begin{vmatrix} s+1 & 1 & 0 \\ e^s & e^s & e^s \\ e^{-s} & -e^{-s} & e^{-s} \end{vmatrix} = 2(s+1), \tag{8}$$

$$\tilde{W}[\boldsymbol{u}](s,x) = \begin{vmatrix} s+1 & 1 & x+1 \\ e^s & e^s & e^x \\ e^{-s} & -e^{-s} & e^{-x} \end{vmatrix} = se^{s-x} + (s+2)e^{x-s} - 2(x+1), \tag{9}$$

$$H_\theta(x,s) = \theta(x-s)(se^{s-x} + (s+2)e^{x-s} - 2x - 2)/(2s+2). \tag{10}$$

Then Green's function exists for problem with BCs $u(0) = u'(0) = u(1) = 0$ ($D(\boldsymbol{L})[\boldsymbol{u}] = 2(e-2)$) and from Theorem 1 we have

$$\begin{aligned} G^{\mathrm{cl}}(x,s) &= -\tfrac{e^x-x-1}{e-2} H_\theta(1,s) + H_\theta(x,s) \\ &= -\tfrac{(e^x-x-1)(se^{s-1}+(s+2)e^{1-s}-4)}{2(e-2)(s+1)} + \theta(x-s)\tfrac{se^{s-x}+(s+2)e^{x-s}-2(x+1)}{2(s+1)}. \end{aligned}$$

Now we express Green's function for problem with NBCs via Green's function $G^{\mathrm{cl}}(x,s)$. For problem with NBCs (7) Green's function exists if $D(\boldsymbol{L})[\boldsymbol{u}] = 2e - 4 - 2\gamma(e^\xi - \xi - 1) \neq 0$. So, for $\gamma \neq (e-2)/(e^\xi - 1 - \xi)$ Green's function

$$\begin{aligned} G(x,s) &= G^{\mathrm{cl}}(x,s) + \tfrac{(e^x-x-1)}{e-2-\gamma(e^\xi-\xi-1)}\gamma G^{\mathrm{cl}}(\xi,s) = \gamma\tfrac{(e^x-x-1)}{e-2-\gamma(e^\xi-\xi-1)} \\ &\quad\times\Big(-\tfrac{(e^\xi-\xi-1)(se^{s-1}+(s+2)e^{1-s}-4)}{2(e-2)(s+1)} + \theta(\xi-s)\tfrac{se^{s-\xi}+(s+2)e^{\xi-s}-2(\xi+1)}{2(s+1)}\Big) \\ &\quad- \tfrac{(e^x-x-1)(se^{s-1}+(s+2)e^{1-s}-4)}{2(e-2)(s+1)} + \theta(x-s)\tfrac{se^{s-x}+(s+2)e^{x-s}-2(x+1)}{2(s+1)}. \end{aligned}$$

Example 2. We consider second-order problem with two NBCs

$$u'' + \frac{2x}{1+x^2}u' = f, \quad x \in (0,1), \tag{11}$$

$$u(0) = \delta u(\xi), \quad \xi \in (0,1), \qquad u(1) = \gamma u(\nu), \quad \nu \in (0,1). \tag{12}$$

Fundamental system of homogeneous equation (11) is $u_1(x) = 4/\pi$, $u_2(x) = \arctan x$. Green's function for problem with IC $u(0) = u'(0) = 0$ is equal to

$$H_\theta(x,s) = \theta(x-s)(\arctan x - \arctan s)(1+s^2) = \theta(x-s)\arctan\frac{x-s}{1+xs}(1+s^2).$$

Green's function for differential equation (11) with Dirichlet BCs $u(0) = u(1) = 0$ has the following form

$$G_{\mathrm{D}}(x,s) = H_\theta(x,s) - \frac{4}{\pi}\arctan x \cdot H_\theta(1,s). \tag{13}$$

Green's function for problem with NBCs (12) exists if $\vartheta = D(\boldsymbol{L})[\boldsymbol{u}] = (1-\delta) - \gamma(1-\delta)\frac{4}{\pi}\arctan\nu + \delta(1-\gamma)\frac{4}{\pi}\arctan\xi \neq 0$ and

$$
\begin{aligned}
G(x,s) &= \frac{1}{\vartheta}\begin{vmatrix} (1-\delta)\frac{4}{\pi} & (1-\gamma)\frac{4}{\pi} & \frac{4}{\pi} \\ -\delta\arctan\xi & \frac{\pi}{4}-\gamma\arctan\nu & \arctan x \\ -\delta G_{\mathrm{D}}(\xi,s) & -\gamma G_{\mathrm{D}}(\nu,s) & G_{\mathrm{D}}(x,s)\end{vmatrix} \\
&= G_{\mathrm{D}}(x,s) - \delta G_{\mathrm{D}}(\xi,s)\biggl((1-\gamma)\frac{4}{\pi}\arctan x - 1 + \frac{4}{\pi}\gamma\arctan\nu\biggr)/\vartheta \\
&\quad + \gamma G_{\mathrm{D}}(\nu,s)\biggl((1-\delta)\frac{4}{\pi}\arctan x + \frac{4}{\pi}\delta\arctan\xi\biggr)/\vartheta. \qquad (14)
\end{aligned}
$$

We can use Theorem 1 directly and get

$$
\begin{aligned}
G(x,s) &= H_\theta(x,s) - \delta H_\theta(\xi,s)\biggl((1-\gamma)\frac{4}{\pi}\arctan x - 1 + \frac{4}{\pi}\gamma\arctan\nu\biggr)/\vartheta \\
&\quad - \bigl(H_\theta(1,s) - \gamma H_\theta(\nu,s)\bigr)\biggl((1-\delta)\frac{4}{\pi}\arctan x + \frac{4}{\pi}\delta\arctan\xi\biggr)/\vartheta. \qquad (15)
\end{aligned}
$$

If we substitute H_θ into (15) then get explicit expresion for Green's functon.

For example, NBCs $u(0)=0$, $u(1)=2u(\sqrt{3}/3)$ are special case of BC (12) ($\delta=0$, $\gamma=2$, $\nu=\sqrt{3}/3$). In this case

$$
\begin{aligned}
G(x,s) &= 3(1+s^2)\arctan x\biggl(1-\frac{4}{\pi}\arctan s\biggr) \\
&\quad + (1+s^2)\begin{cases}\arctan x - \arctan s, & x>s,\\ 0, & x\leqslant s,\end{cases} \\
&\quad - 6(1+s^2)\arctan x\begin{cases}\frac{4}{6}-\frac{4}{\pi}\arctan s, & s<\sqrt{3}/3,\\ 0, & s\geqslant\sqrt{3}/3.\end{cases}
\end{aligned}
$$

Example 3. Let us consider third-order problem

$$
u''' + \frac{3x}{1+x^2}u'' = f, \quad x\in(0,1) \qquad (16)
$$

$$
u(0)=0, \qquad u'(0)=\gamma u(\nu), \quad \nu\in(0,1), \qquad u''(1)=\beta u(\eta), \quad \eta\in(0,1). \qquad (17)
$$

The fundamental system for this problem is $\{1, x, \sqrt{1+x^2}\}$.

Green's function for differential equation (16) with ICs $u(0)=u'(0)=u''(0)=0$ is equal to

$$
H_\theta(x,s) = \theta(x-s)\bigl(\sqrt{(1+x^2)(1+s^2)} - 1 - xs\bigr)(1+s^2).
$$

Then Green's function for BC $u(0)=u'(0)=u''(1)=0$ ($D(\boldsymbol{L})[\boldsymbol{u}]=\sqrt{2}/4$) is

$$
\begin{aligned}
G^{\mathrm{cl}}(x,s) &= H_\theta(x,s) - 2\sqrt{2}\bigl(\sqrt{1+x^2}-1\bigr)H''_\theta(1,s) \\
&= \theta(x-s)\bigl(\sqrt{(1+x^2)(1+s^2)} - 1 - xs\bigr)(1+s^2) \\
&\quad - \bigl(\sqrt{1+x^2}-1\bigr)(1+s^2)\sqrt{1+s^2}.
\end{aligned}
$$

Green's function for problem (16) with NBCs (17) exists for $\vartheta = D(\boldsymbol{L})[\boldsymbol{u}] = (1 - \gamma\nu)(\frac{1}{2\sqrt{2}} + \beta - \beta\sqrt{1+\eta^2}) + \gamma\beta\eta(1 - \sqrt{1+\nu^2}) \neq 0$ and

$$G(x,s) = G^{\mathrm{cl}}(x,s) - \beta G^{\mathrm{cl}}(\eta,s)\big((1-\nu\gamma)(1-\sqrt{1+x^2}) + x\gamma(1-\sqrt{1+\nu^2})\big)/\vartheta$$
$$- \gamma G^{\mathrm{cl}}(\nu,s)\big(\beta\eta(1-\sqrt{1+x^2}) - x(\beta(1-\sqrt{1+\eta^2}) + \sqrt{2}/4)\big)/\vartheta.$$

Example 4. Let us consider fourth-order problem

$$u^{(4)} - \frac{4}{(x+1)^2}u'' + \frac{8}{(x+1)^3}u' - \frac{8}{(x+1)^4}u = f, \quad x \in (0,1), \tag{18}$$

$$u(0) = 0, \qquad u'(0) = 0, \qquad u''(0) = 0, \qquad u(1) = \beta u(\eta), \quad \eta \in (0,1). \tag{19}$$

For Eq. (18) we take the fundamental system $u_1(x) = x+1$, $u_2(x) = (x+1)^2$, $u_3(x) = (x+1)^4$, $u_4(x) = (x+1)^{-1}$. Then

$$\begin{aligned}
\tilde{W}[\boldsymbol{u}](s,x) &= \begin{vmatrix} s+1 & 1 & 0 & u_1(x) \\ (s+1)^2 & 2(s+1) & 2 & u_2(x) \\ (s+1)^4 & 4(s+1)^3 & 12(s+1)^2 & u_3(x) \\ (s+1)^{-1} & -(s+1)^{-2} & 2(s+1)^{-3} & u_4(x) \end{vmatrix} \\
&= -30(s+1)^2u_1(x) + 30(s+1)u_2(x) - 6(s+1)^{-1}u_3(x) \\
&\quad + 6(s+1)^4u_4(x), \\
W[\boldsymbol{u}](s) &= -30(s+1)^2\cdot 0 + 30(s+1)\cdot 0 - 6(s+1)^{-1}\cdot 24(s+1) \\
&\quad + 6(s+1)^4\cdot(-6)(s+1)^{-4} = -180, \\
H_\theta(x,s) &= -\theta(x-s)\tilde{W}[\boldsymbol{u}](s,x)/180.
\end{aligned}$$

Green's function for problem with NBCs (18), (19) exists if $\vartheta(\beta) = D(\boldsymbol{L})[\boldsymbol{u}] = -33 - \beta\tilde{W}(\boldsymbol{L})[\boldsymbol{u}](0,\eta) = -33 + \beta(30(\eta+1) - 30(\eta+1)^2 + 6(\eta+1)^4 - 6(\eta+1)^{-1}) \neq 0$.

Green's function for problem with BCs $u(0) = u'(0) = u''(0) = u(1) = 0$ (the case $\beta = 0$) is equal to

$$\bar{G}(x,s) = -\theta(x-s)\tilde{W}[\boldsymbol{u}](s,x)/180 + \tilde{W}[\boldsymbol{u}](0,x)\cdot\tilde{W}[\boldsymbol{u}](s,1)/\vartheta(0)/180.$$

Green's function for problem with NBCs is

$$G(x,s) = \bar{G}(x,s) + \beta\bar{G}(\eta,s)\cdot\tilde{W}[\boldsymbol{u}](0,x)/\vartheta(\beta).$$

These examples show how we can construct Green's function for problem with NBCs if we know fundamental basis of homogeneous differential equation and Green's function for problem with simpler conditions.

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REZIUMĖ

Kraštiniai uždaviniai su nelokaliosiomis kraštinėmis sąlygomis ir šių uždavinių Gryno funkcijos

S. Roman, A. Štikonas

Šiame darbe nagrinėjama n-tos eilės tiesinė diferencialinė lygtis su n papildomų sąlygų. Pateikta Gryno funkcijos formulė bei jos egzistavimo sąlygos. Nagrinėjamo uždavinio Gryno funkcija išreiškiama per kito analogiško uždavinio Gryno funkciją. Tai leidžia lengvai rasti Gryno funkciją uždaviniams su nelokaliosiomis kraštinėmis sąlygomis.

Raktiniai žodžiai: Gryno funkcija, nelokaliosios kraštinės sąlygos.