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Numerical investigation of alternating-direction method for Poisson equation with weighted integral conditions

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Abstract. The present paper deals with a generalization of the alternating-direction implicit (ADI) method for a two dimensional Poisson equation in a rectangle domain with a weighted integral boundary condition in one coordinate direction. We consider the alternating direction method for a system of difference equations that approximates Poisson equation with weighed integral boundary conditions with the fourth-order accuracy. Sufficient conditions of stability for ADI method are investigated numerically. An analysis of results of computational experiments is presented.

Keywords: elliptic equation, nonlocal integral conditions, finite-difference method, alternating-direction method, convergence of iterative method.

Introduction

Various numerical finite difference schemes have been proposed to solve boundary value problems of elliptic partial differential equations. The demand for better and more efficient methods has grown as the range of applications has increased. Mathematical models involving elliptic partial differential equations arise in diverse applications such as heat conduction, electrostatics, mechanical engineering and theoretical physics [1]. The simplest discretization method for elliptic equations is the classical finite difference method then equation are discretized and reduced to finite difference equations, and then linear systems with large sparse matrices are solved using appropriate numerical techniques. But many difficulties arise in the case of more complicated differential operators, boundary conditions, and computational domains. Problems with nonlocal boundary conditions have been intensively studied during last time. Second-order finite-difference methods for elliptic equations with nonlocal conditions were considered in [3, 6, 10], where the main aspect was the convergence of methods. In [7, 5], a difference scheme of fourth-order accuracy was considered for elliptic equation with nonlocal integral conditions. Necessary and sufficient conditions and a convergent iterative solution method were found for the system of difference equations in the case of a unique solution. In this paper, we consider the alternating direction method for a system of difference equations that approximates Poisson equation with weighed integral boundary conditions with the fourth-order accuracy.

The matrix of a system of difference equations with nonlocal conditions is always nonsymmetric, and its spectrum can be rather complicated, i.e., there may exist negative, multiple, or complex eigenvalues [8]. The purpose of the present work is to perform a numerical investigation of the influence of the integral boundary conditions on the ADI method applicability to solution of the elliptic differential equations with nonlocal conditions [5].

This paper is organized as follows. In Section 1, we formulate a difference problem and write the fourth-order ADI method [4]. In Section 2, we present results of numerical experiment. Section 3 contains some brief conclusions and comments.

1 Statement of a difference problem

We consider two-dimensional elliptic equation with weighted integral conditions:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

$$u(0,y) = \gamma_0 \int_0^{L_x} \alpha(x,y) u(x,y)\,dx + v^l(y), \quad y \in [0, L_y], \tag{2}$$

$$u(L_x,y) = \gamma_1 \int_0^{L_x} \beta(x,y) u(x,y)\,dx + v^r(y), \quad y \in [0, L_y], \tag{3}$$

$$u(x,0) = w^l(x), \qquad u(x,L_y) = w^r(x), \quad x \in [0, L_x]. \tag{4}$$

where $\Omega = (0, L_x) \times (0, L_y)$ is a rectangular domain, γ_0 and γ_1 are given constants. Finite-difference methods of fourth order accuracy for elliptic equations with nonlocal conditions ($\alpha(x,y) = \beta(x,y) = 1$) were considered and investigated in [7]. The ADI method for this problem was considered in [5, 9].

In the domain $\overline{\Omega}$ we consider the grid $\overline{\omega}^h = \overline{\omega}^h_x \times \overline{\omega}^h_y$ with steps $h_x = L_x/n$ and $h_y = L_y/m$.

Equations (1)–(4) are replaced with a finite-difference equations:

$$\begin{gathered} -\left(\delta_x^2 + \delta_y^2 + \frac{h_x^2 + h_y^2}{12}\delta_x^2\delta_y^2\right) u_{ij} = \overline{f}_{ij}, \\ \overline{f}_{ij} = f_{ij} + \tfrac{h_x^2}{12}\delta_x^2 f_{ij} + \frac{h_y^2}{12}\delta_y^2 f_{ij}, \quad (x_i, y_i) \in \omega^h, \end{gathered} \tag{5}$$

$$u_{0j} = \gamma_0 \sum_{i=0}^{n} \alpha_{ij} u_{ij} \rho_i h_x + v^l_j, \quad y_j \in \overline{\omega}^h_y, \tag{6}$$

$$u_{nj} = \gamma_1 \sum_{i=0}^{n} \beta_{ij} u_{ij} \rho_i h_x + v^r_j, \quad y_j \in \overline{\omega}^h_y, \tag{7}$$

$$u_{i0} = w^l_i, \qquad u_{im} = w^r_i, \quad x_i \in \overline{\omega}^h_x. \tag{8}$$

where δ_x^2 and δ_y^2 are second order central difference operators. We assume the compartability of the boundary conditions

$$w^l_0 = \gamma_0 \sum_{i=0}^{n} \alpha_{i0} w^l_i \rho_i h_x + v^l_0, \qquad w^r_n = \gamma_1 \sum_{i=0}^{n} \beta_{in} w^r_i \rho_i h_x + v^r_n.$$

We approximate integral conditions (2)–(4) using the Simpson formula with the fourth order approximation error: $\rho_i = (3 + (-1)^{i-1})/3$, $i = 1, \dots, n-1$, $\rho_0 = \rho_n = 1/3$, n is even.

We consider the corresponding discrete Sturm–Liouville problem

$$-\delta_x^2 v_i = \lambda v_i, \quad x_i \in \omega_x^h, \tag{9}$$

$$v_{0j} = \gamma_0 \sum_{i=0}^{n} \alpha_i v_i \rho_i h_x, \qquad v_{nj} = \gamma_1 \sum_{i=0}^{n} \beta_i v_i \rho_i h_x, \tag{10}$$

where α_i, β_i, $\rho_i \in F(\overline{\omega}_x^h)$. Problem (9)–(10) can be interpetated as difference analogue of differential eigenvalue problem:

$$-\frac{d^2 v}{dx^2} = \lambda v, \quad 0 < x < L_x, \tag{11}$$

$$v(0) = \gamma_0 \int_0^{L_x} \alpha_0(x) v(x)\, dx, \qquad v(L_x) = \gamma_1 \int_0^{L_x} \alpha_1(x) v(x)\, dx. \tag{12}$$

Stationary problems with different nonlocal conditions are investigated in paper [2] and the following theorem is valid.

Theorem 1. $\lambda = 0$ *is eigenvalue of matrix* $-\Lambda_x$ *if and only if*

$$\det \mathbf{K} \cdot \gamma_0 \gamma_1 - k_{00}\gamma_0 - k_{11}\gamma_1 + 1 = 0, \tag{13}$$

where

$$\mathbf{K} = \begin{pmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{pmatrix}, \qquad k_{i0} = \int_0^{L_x} \alpha_i(x) x\, dx, \qquad k_{i1} = \int_0^{L_x} \alpha_i(x)(1-x)\, dx, \quad i = 0, 1.$$

Namely, $\lambda = 0$ is eigenvalue of matrix $-\Lambda_x$ if and only if point (γ_0, γ_1) belongs to the second order curve.

- If $\det \mathbf{K} \neq 0$, $k_{01}k_{10} \neq 0$ it is hyperbola with vertical and horizontal asymptotes;
- If $\det \mathbf{K} \neq 0$, $k_{01}k_{10} = 0$ it is a pair of perpendicular lines (the vertical line and the horizontal line);
- If $\det \mathbf{K} = 0$, $k_{01}k_{10} \neq 0$ it is a line.

2 Numerical experiment

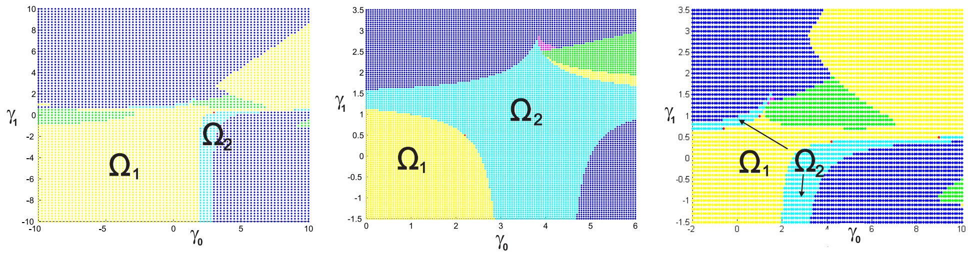
We consider a problem (1)–(4) in unit square domain $[0, 1] \times [0, 1]$ with weights

$$\alpha(x) = \frac{2(1 - bx)}{2 - b}, \qquad \beta(x) = \frac{2(1 + bx)}{2 + b}.$$

This gives a qualitatively different regions of convergence for the ADI method.

The exact solution of this test problem is given by

$$u(x, y) = x^6 + y^6.$$



(1) $b = -3$. (2) $b = -3.5$. (3) $b = -2.5$.

Fig. 1. Domains of the stability.

The right-hand side function $f(x, y)$, initial and boundary conditions were prescribed to satisfy the given exact solution $u(x, y)$. We consider uniform grids with different mesh sizes and analyze the convergence and accuracy of the computed solution from the present ADI scheme. We compute the maximum norm of the error of the numerical solution with respect to the exact solution, which is defined as

$$\|\varepsilon\|_{C,h} = \max_{j=1,\dots,m} \max_{i=1,\dots,n} \left|u(x_i, y_j) - u_{ij}\right|.$$

Test problems were solved with different values of parameters b, γ_1 and γ_2.

Numerical experiments let us to determine domains where real parts of all eigenvalues are positive and domains where they are negative or zero. We fix values of the parameter b ($b = -2.5$, $b = -3$, or $b = -3.5$) and study the influence of weight functions on the stability of solutions given by ADI method depending on γ_1 and γ_2. In all cases there exists a region where ADI method gives stable solutions. Fig. 1 demonstrate the regions where the real part of eigenvalues for operator of the ADI method are positive or negative for different weights α and β. All eigenvalues are positive in Ω_1, so ADI method is converged. There exists negative eigenvalue in Ω_2, but ADI method is converged in Ω_2, too. ADI method isn't converged in other parts of domain Ω. Therefore, Ω_1 is the domain of the stability of the solution finding by ADI method according to theoretical result based on spectral stability while Ω_2 is the region of the stability of the solution established by numerical experiment.

In the case $b = -3.5$ according to theoretical results connected with eigenvalue problem stability region for ADI method is region Ω_1 under the hyperbola in the third quarter (Fig. 1(2)). So convergence region for problem with $b = -3.5$ where positive eigenvalues exist located under the hyperbola, but the method converges in a much larger area with a negative eigenvalue. Table 1 presents the performances of the algorithm for various weights depends on $b = -3.5$. Errors for the discrete solution on the grids 32×32, minimal eigenvalues, convergence rates and number of iterations are presented for different γ_1, γ_2. The results show that the method can be applied even when there is a negative or zero eigenvalue. The method has the fourth-order convergence.

For $b = -3$ stability region of positive eigenvalues is angle between a pair of perpendicular lines. (Fig. 1(1)).

For $b = -2.5$ region of positive eigenvalues is located between a pair of hyperbolas (Fig. 1(3)). Table 2 presents the performances of the algorithm for various weights depends on $b = -2.5$. Note that for large values of γ_0, γ_1 error increases.

Table 1. Errors for the discrete solution for problem solved with different γ_1, γ_2 for $b = -3.5$.

γ_0	γ_1	h	$\lambda_{\min}$	ε_k	$\varepsilon_{k-1}/\varepsilon_k$	Number of iter.
1.0	1.0	0.125	0.000	0.002822		18
		0.0625		$1.8071 \cdot 10^{-4}$	15.61	23
		0.03125		$1.1363 \cdot 10^{-5}$	15.90	27
		0.015625		$7.1129 \cdot 10^{-7}$	15.97	31
10.0	0.0	0.125	-7.80	0.01918		23
		0.0625		$1.2358 \cdot 10^{-3}$	15.52	28
		0.03125		$7.7792 \cdot 10^{-5}$	15.88	32
		0.015625		$4.8705 \cdot 10^{-6}$	15.97	36
4.0	2.0	0.125	-3.91	0.03858		20
		0.0625		$2.4979 \cdot 10^{-3}$	15.44	24
		0.03125		$1.5729 \cdot 10^{-4}$	15.88	29
		0.015625		$9.8482 \cdot 10^{-6}$	15.97	33

Table 2. Results for $b = -2.5$.

γ_0	γ_1	h	$\lambda_{\min}$	ε_k	$\varepsilon_{k-1}/\varepsilon_k$	Number of iter.
4.5	0.3	0.125	-3.1618	0.0095502		19
		0.0625	-3.1650	$6.1406 \cdot 10^{-4}$	15.5525	24
		0.03125	-3.1652	$3.8637 \cdot 10^{-5}$	15.8930	28
		0.015625	-3.1653	$2.4188 \cdot 10^{-6}$	15.9736	33
0.0	0.95	0.125	-8.0803	0.06771		23
		0.0625	-8.0594	$4.2881 \cdot 10^{-3}$	15.7895	28
		0.03125	-8.0581	$2.6968 \cdot 10^{-4}$	15.9006	32
		0.015625	-8.0580	$1.6885 \cdot 10^{-5}$	15.9721	37
-10	-10	0.125	32.7539	0.0025521		14
		0.0625	32.7801	0.0002651	9.62608	18
		0.03125	32.7818	$2.6459 \cdot 10^{-5}$	10.0200	22
		0.015625	32.7819	$2.4178 \cdot 10^{-6}$	10.9434	27
2	0.5	0.125	4.4570	0.0027356		17
		0.0625	4.4597	$1.7531 \cdot 10^{-4}$	15.6044	21
		0.03125	4.4599	$1.1024 \cdot 10^{-5}$	15.9024	26
		0.015625	4.4599	$6.8999 \cdot 10^{-7}$	15.9773	30
10	1	0.1250	37.6440	0.0060		13
		0.0625	37.6874	$3.9819 \cdot 10^{-4}$	15.0682	18
		0.03125	37.6885	$3.9388 \cdot 10^{-5}$	10.1094	22
		0.0156	37.6902	$2.7557 \cdot 10^{-4}$	0.1429	26

3 Conclusions and remarks

The ADI method can be used for the Poisson equation with weighted integral condition with nontrivial weights $\alpha(x)$ and $\beta(x)$. Nonlocal integral conditions never make more problems than classical conditions both in number of iterations and precision of solution. But this conditions affect the region of convergence of the method. Convergence domain depends essentially on the weight functions. The values of parameters γ_1 and γ_2 in NBC are essential for the stability of the ADI method. The results of numerical experiment are in good agreement with existing theoretical results for two dimensional Poisson equation in a rectangle domain with a weighted integral boundary condition in one coordinate direction [5]. But numerical experiments show that this problem is complicated and additional investigation is needed.

References

[1] G. Haase C.C. Douglas and U. Langer. *A Tutorial on Elliptic PDE Solvers and Their Parallelization.* SIAM Series Software, Environments, and Tools, 2003.

[2] R. Čiegis, A. Štikonas, O. Štikonienė and O. Subač. A monotonic finite-diference scheme for a parabolic problem with nonlocal conditions. *Differ. Equat.*, **38**(7):1027–1037, 2002.

[3] V.A. Il'in and E.I. Moiseev. Two-dimensional nonlocal boundary value problem for Poisson's operator in differential and difference variants. *Mat. Model.*, **2**(8):132–156, 1990 (in Russian).

[4] A.A. Samarskii and E.S. Nikolaev. *The Methods for Solution of Difference Equations.* Moscow, Nauka, 1978 (in Russian).

[5] M. Sapagovas and O. Štikonienė. A fourth-order alternating direction method for difference schemes with nonlocal condition. *Lith. Math. J.*, **49**(3):309–317, 2009.

[6] M.P. Sapagovas. Difference scheme for two-dimensional elliptic problem with an integral condition. *Lith. Math. J.*, **23**(3):155–159, 1983.

[7] M.P. Sapagovas. Difference method on increased order of accuracy for the Poisson equation with nonlocal conditions. *Differ. Equ.*, **44**(7):988–998, 2008.

[8] M.P. Sapagovas and A.D. Štikonas. On the structure of the spectrum of a differential operator with a nonlocal condition. *Differ. Equ.*, **41**(7):961–969, 2005.

[9] O. Štikonienė. Numerical investigation of fourth-order alternating direction method for Poisson equation with integral conditions. In *Proceeding of Intern. Conf. Differ. Equations and their Applic. DETA-2009*, pp. 139–146, Kaunas, 2009.

[10] Y. Wang. Solutions to nonlinear elliptic equations with a nonlocal boundary condition. *Electron. J. Differ. Equ.*, **2002**(2):1–16, 2002.

REZIUMĖ

Kintamųjų krypčių metodo ketvirtos eilės tikslumo Puasono baigtinių skirtumų lygčiai su integralinėmis sąlygomis skaitinė analizė

O. Štikonienė, M. Sapagovas

Straipsnyje išnagrinėtas kintamųjų krypčių metodo apibendrinimas dvimatei Puasono lygčiai stačiakampėje srityje su svorinėmis integralinėmis kraštinėmis sąlygomis pagal vieną kryptį. Pakankamos kintamųjų krypčių metodo stabilumo sąlygos tiriamos skaitiškai.

Raktiniai žodžiai: elipsinė lygtis, nelokaliosios integralinės sąlygos, kintamųjų krypčių metodas, konvergavimas