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On the periodic zeta-function with rational parameter

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Abstract. We obtain an asymptotic formula with estimated error term for the fourth power moment of the periodic zeta-function with rational parameter.

Keywords: approximate functional equation, periodic zeta-function, Riemann zeta-function.

Let $s=\sigma+it$ and $\lambda\in\mathbb{R}$. The periodic zeta-function $\zeta_\lambda(s)$, with parameter λ is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta_\lambda(s)=\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{m^s},$$

and, in the case $0<\lambda<1$, is analytically continued to entire function. In [3], we considered the case of irrational parameter λ and obtained the following asymptotic formula for the fourth power moment of $\zeta_\lambda(s)$. Let $\frac{1}{2}<\sigma<1$ and $T\to\infty$. Then for every $\varepsilon>0$,

$$\int_1^T\bigl|\zeta_\lambda(\sigma+it)\bigr|^4dt \\ =T\biggl(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}-2\sum_{m_1n_1=m_2n_2}\frac{\sin^2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}}\biggr)+O\bigl(T^{\frac{3}{2}-\sigma+\varepsilon}\bigr).$$

Here $\zeta(s)$ denotes the Riemann zeta-function. The aim of this note is to obtain a similar formula for the rational parameter λ. We notice that the case of rational λ is more complicated, and the formula obtained is valid in a more narrow region than that in the case of irrational λ.

Theorem 1. *Suppose that the number λ is rational, $0<\lambda<1$, $\frac{3}{4}<\sigma<1$ and $T\to\infty$. Then, for every $\varepsilon>0$,*

$$\int_1^T\bigl|\zeta_\lambda(\sigma+it)\bigr|^4dt \\ =T\biggl(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}-2\sum_{m_1n_1=m_2n_2}\frac{\sin^2\pi\lambda(m_1+n_1-m_2-n_2)}{(m_1n_1)^{2\sigma}}\biggr)+O\bigl(T^{\frac{7}{4}-\sigma+\varepsilon}\bigr).$$

For the proof of Theorem 1, we apply, as in the case of irrational λ, an approximate functional equation for $\zeta_\lambda(s)$. Let $[u]$ denote the integer part of u. Define the functions

$$p(t) = \left[\sqrt{\frac{t}{2\pi}} - 1\right], \qquad q(t) = \left[\sqrt{\frac{t}{2\pi}}\right], \qquad g(\lambda, t) = 2\sqrt{\frac{t}{2\pi}} - p(t) - q(t) - \lambda - 1,$$

$$\begin{aligned} f(\lambda, t) = &- \frac{t}{2\pi}\log\frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(1-\lambda^2) + p(t) - q(t) \\ &+ 2\sqrt{\frac{t}{2\pi}}\big(q(t) - p(t) + \lambda - 1\big) - \frac{1}{2}\big(q(t) + p(t)\big) - \lambda\big(1 + q(t) - p(t)\big) \end{aligned}$$

and

$$\psi(t) = \frac{\cos\pi\left(\frac{t^2}{2} - t - \frac{1}{8}\right)}{\cos\pi t}.$$

Lemma 1. *Suppose that* $0 < \lambda < 1$, $0 \leqslant \sigma \leqslant 1$ *and* $t \geqslant t_0 > 0$. *Then*

$$\begin{aligned} \zeta_\lambda(s) = &\sum_{1 \leqslant m \leqslant p(t)} \frac{e^{2\pi i\lambda m}}{m^s} + \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{it+\frac{\pi i}{4}} \sum_{0 \leqslant m \leqslant q(t)} \frac{1}{(m+\lambda)^{1-s}} \\ &+ \left(\frac{t}{2\pi}\right)^{-\frac{\sigma}{2}} e^{\pi i f(\lambda,t) + 2\pi i\lambda} \psi\big(g(\lambda, t)\big) + O\big(t^{\frac{\sigma}{2}-1}\big). \end{aligned}$$

The lemma is a special case of the approximate functional equation obtained in [1] for the Lerch zeta-function.

Denote

$$S_1(s) = \sum_{1 \leqslant m \leqslant p(t)} \frac{e^{2\pi i\lambda m}}{m^s}, \qquad S_2(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{it+\frac{\pi i}{4}} \sum_{0 \leqslant m \leqslant q(t)} \frac{1}{(m+\lambda)^{1-s}}.$$

Then Lemma 1 implies, for $0 \leqslant \sigma \leqslant 1$, the estimate

$$\zeta_\lambda(s) = S_1(s) + S_2(s) + O\big(t^{-\frac{1}{4}}\big). \tag{1}$$

Therefore, for the proof of Theorem 1, it suffices to find the fourth power moments for the sums $S_1(s)$ and $S_2(s)$. We observe that the fourth moment of $S_1(s)$ does not depend of the arithmetical nature of the parameter λ. Therefore, we have the following statement.

Lemma 2. *Suppose that λ is irrational,* $0 < \lambda < 1$, $\frac{1}{2} < \sigma < 1$ *and* $T \to \infty$. *Then, for every* $\varepsilon > 0$,

$$\begin{aligned} &\int_1^T \big|S_1(\sigma + it)\big|^4 dt \\ &\quad = T\left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2\sum_{m_1 n_1 = m_2 n_2} \frac{\sin^2 \pi\lambda(m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}}\right) + O\big(T^{\frac{3}{2}-\sigma+\varepsilon}\big). \end{aligned}$$

Proof of the lemma is given in [3].

Lemma 3. *Suppose that the number λ, $0 < \lambda < 1$, is rational, $\frac{3}{4} < \sigma < 1$ and $T \to \infty$. Then, for every $\varepsilon > 0$,*

$$\int_1^T \left|S_2(\sigma + it)\right|^4 dt = O(T^{\frac{5}{2} - 2\sigma + \varepsilon}).$$

Proof. Consider the fourth moment

$$J(T) = \int_1^T \left| \sum_{0 \leqslant m \leqslant q(t)} \frac{1}{(m+\lambda)^{1-\sigma-it}} \right|^4 dt.$$

Clearly, we have that

$$J(T) = J_1(T) + O_\lambda(T), \tag{2}$$

where

$$J_1(T) = \int_1^T \left| \sum_{1 \leqslant m \leqslant q(t)} \frac{1}{(m+\lambda)^{1-\sigma-it}} \right|^4 dt.$$

It is not difficult to see that

$$J_1(T) = \int_1^T \sum_{1 \leqslant m_1, n_1, m_2, n_2 \leqslant q(t)} \frac{1}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}} \times \left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right)^{it} dt.$$

Let $T_1 = 2\pi \max(m_1^2, n_1^2, m_2^2, n_2^2)$. Then

$$J_1(T) = \sum_{1 \leqslant m_1, n_1, m_2, n_2 \leqslant q(t)} \frac{1}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}} \times \int_{T_1}^T \left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right)^{it} dt. \tag{3}$$

We investigate two cases: $(m_1+\lambda)(n_1+\lambda) = (m_2+\lambda)(n_2+\lambda)$ and $(m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)$. In the first case, the integral in (3) is equal to $T - T_1$, in the second case, we find after integration that, for that integral, the estimate

$$O\left(\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right|^{-1} \right)$$

holds. The above remarks together with (3) show that

$$J_1(T) \ll \sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda) = (m_2+\lambda)(n_2+\lambda)} \frac{T - T_1}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} + \sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)} \frac{\left|\log\left(\frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)}\right)\right|^{-1}}{((m_1+\lambda)(n_1+\lambda)(m_2+\lambda)(n_2+\lambda))^{1-\sigma}}, \tag{4}$$

where the star "$*$" means that each m_1, n_1, m_2, n_2 runs over $[1, q(T)]$. Denoting by $d(m)$ the number of divisors of m, we have that

$$
\begin{aligned}
&T \sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)} \frac{1}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} \\
&\quad \ll T \sum_{m \leqslant q^2(T)} \sum_{m-\sqrt{T} \leqslant n \leqslant m+\sqrt{T}} \frac{d(m)d(n)}{m^{2-2\sigma}} \\
&\quad \ll T^{\frac{3}{2}+\varepsilon} \sum_{m \leqslant q^2(T)} \frac{d(m)}{m^{2-2\sigma}} \ll T^{\frac{1}{2}+2\sigma+\varepsilon}. \qquad (5)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda)=(m_2+\lambda)(n_2+\lambda)} \frac{T_1}{((m_1+\lambda)(n_1+\lambda))^{2-2\sigma}} \\
&\quad \ll \sum_{m_1, n_1 \leqslant q(T)} \sum_{m_1 n_1 - \sqrt{T} \leqslant n \leqslant m_1 n_1 + \sqrt{T}} \frac{m_1^2 d(m_1 n_1) d(m)}{(m_1 n_1)^{2-2\sigma}} \\
&\quad \ll T^{\frac{1}{2}+\varepsilon} \Bigg(\sum_{m_1 \leqslant q(T)} m_1^{2\sigma} \sum_{n_1 \leqslant q(T)} \frac{1}{n_1^{2-2\sigma}} \Bigg) \ll T^{\frac{1}{2}+2\sigma+\varepsilon}. \qquad (6)
\end{aligned}
$$

The inequality

$$
\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right| \geqslant c(\lambda) \min \left(\frac{1}{(m_1+\lambda)(n_1+\lambda)}, \frac{1}{(m_2+\lambda)(n_2+\lambda)} \right),
$$

with a certain $c(\lambda) > 0$ and application of the Montgomery–Vaughan theorem [2] yield

$$
\sideset{}{^*}\sum_{(m_1+\lambda)(n_1+\lambda) \neq (m_2+\lambda)(n_2+\lambda)} \frac{\left| \log \frac{(m_1+\lambda)(n_1+\lambda)}{(m_2+\lambda)(n_2+\lambda)} \right|^{-1}}{(m_1 n_1 m_2 n_2)^{1-\sigma}} \ll_\lambda T^{\varepsilon} \sum_{1 \leqslant m \leqslant q^2(T)} \frac{m}{m^{2-2\sigma}} \ll_\lambda T^{2\sigma+\varepsilon}.
$$

This and (4)–(6) give the estimate

$$
J_1(T) \ll_\lambda T^{\frac{1}{2}+2\sigma+\varepsilon}.
$$

and, in view of (1),

$$
J(T) \ll_\lambda T^{\frac{1}{2}+2\sigma+\varepsilon}.
$$

Hence,

$$
\int_1^T \left| S_2(\sigma+it) \right|^4 \ll \int_1^T t^{2-4\sigma}\, dJ(t) \ll_\lambda T^{\frac{5}{2}-2\sigma+\varepsilon}. \qquad \square
$$

Proof of Theorem 1. In view of (1), we have that

$$
\begin{aligned}
\left|\zeta_\lambda(s)\right|^4 =& \left(S_1(s)+S_2(s)+O(t^{-\frac14})\right)^2\left(\overline{S_1(s)}+\overline{S_2(s)}+O\left(t^{-\frac14}\right)\right)^2\\
=&\left|S_1(s)\right|^4+S_1^2(s)\overline{S_2^2(s)}+2S_1^2(s)\overline{S_1(s)S_2(s)}+\overline{S_1^2(s)}S_2^2(s)+\left|S_2(s)\right|^4\\
&+2\overline{S_1(s)}S_2^2(s)\overline{S_2(s)}+2S_1(s)\overline{S_1^2(s)}S_2(s)+2S_1(s)\overline{S_2^2(s)}S_2(s)\\
&+4\left|S_1(s)\right|^2\left|S_2(s)\right|^2+O\left(\left|S_1(s)\right|^3t^{-\frac14}\right)+O\left(\left|S_1(s)\right|^2\left|S_2(s)\right|t^{-\frac14}\right)\\
&+O\left(\left|S_1(s)\right|^2t^{-\frac12}\right)+O\left(\left|S_1(s)\right|\left|S_2(s)\right|^2t^{-\frac14}\right)+O\left(\left|S_2(s)\right|^3t^{-\frac14}\right)\\
&+O\left(\left|S_2(s)\right|^2t^{-\frac12}\right)+O\left(\left|S_1(s)\right|\left|S_2(s)\right|t^{-\frac12}\right)+O\left|S_1(s)\right|t^{-\frac34}\right)\\
&+O\left(\left|S_2(s)\right|t^{-\frac34}\right)+O\left(t^{-1}\right). \qquad \square
\end{aligned}
\tag{7}
$$

By Lemmas 2 and 3, and the Cauchy–Schwarz inequality, we find that

$$
\begin{aligned}
&\int_1^T \left(S_1^2(\sigma+it)\overline{S_2^2(\sigma+it)}+\overline{S_1^2(\sigma+it)}S_2^2(\sigma+it)+4\left|S_1^2(\sigma+it)\right|^2\left|S_2^2(\sigma+it)\right|^2\right)dt\\
&\quad\ll\left(\int_1^T\left|S_1(\sigma+it)\right|^4dt\int_1^T\left|S_2(\sigma+it)\right|^4dt\right)^{\frac12}\ll_\lambda T^{\frac74-\sigma+\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
&\int_1^T \left(2S_1^2(\sigma+it)\overline{S_1(\sigma+it)S_2(\sigma+it)}+2S_1(\sigma+it)\overline{S_1^2(\sigma+it)}S_2(\sigma+it)\right.\\
&\qquad\left.+2\overline{S_1(\sigma+it)}S_2^2(\sigma+it)\overline{S_2(\sigma+it)}+2S_1(\sigma+it)\overline{S_2^2(\sigma+it)}S_2(\sigma+it)\right)dt\\
&\quad\ll\left(\int_1^T\left|S_1(\sigma+it)\right|^2\left|S_2(\sigma+it)\right|^2dt\right)^{\frac12}\\
&\qquad\times\left(\left(\int_1^T\left|S_1(\sigma+it)\right|^4dt\right)^{\frac12}+\left(\int_1^T\left|S_2(\sigma+it)\right|^4dt\right)^{\frac12}\right)\\
&\quad\ll_\lambda\left(\int_1^T\left|S_1(\sigma+it)\right|^4dt\int_1^T\left|S_2(\sigma+it)\right|^4dt\right)^{\frac14}\\
&\qquad\times\left(T^{\frac12}+T^{\frac54-\sigma+\varepsilon}\right)\ll_\lambda\left(T^{\frac78-\frac\sigma2+\varepsilon}\left(T^{\frac12}+T^{\frac54-\sigma+\varepsilon}\right)\right)\ll_\lambda T^{\frac{11}{8}-\frac\sigma2+\varepsilon}.
\end{aligned}
$$

Moreover,

$$
\int_1^T\left|S_1(\sigma+it)\right|^3t^{-\frac14}\,dt\ll_\lambda T^{\frac34+\varepsilon},
$$

$$
\int_1^T\left|S_2(\sigma+it)\right|^3t^{-\frac14}\,dt\ll_\lambda T^{\frac{15}{8}-\frac32\sigma+\varepsilon},
$$

$$
\int_1^T\left|S_1(\sigma+it)\right|\left|S_2(\sigma+it)\right|^2t^{-\frac14}\,dt\ll_\lambda T^{\frac32-\sigma+\varepsilon},
$$

$$\int_1^T \left|S_1(\sigma+it)\right|^2 \left|S_2(\sigma+it)\right| t^{-\frac{1}{4}}\,dt \ll_\lambda T^{\frac{5}{4}-\frac{\sigma}{2}+\varepsilon},$$

$$\int_1^T \left|S_1(\sigma+it)S_2(\sigma+it)\right| t^{-\frac{1}{2}}\,dt \ll_\lambda T^{\frac{7}{8}-\frac{\sigma}{2}+\varepsilon}.$$

These estimates together with Lemmas 2 and 3 prove the theorem.

References

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REZIUMĖ

Periodinės dzeta funkcijos su racionaliuoju parametru
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Straipsnyje gautas liekamojo nario įvertis periodinės dzeta funkcijos su racionaliuoju parametru ketvirtojo momento asimptotinėje formulėje.

Raktiniai žodžiai: artutinė funkcinė lygtis, periodinė dzeta funkcija, Rymano dzeta funkcija.