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On zeros of some composite functions

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Abstract. We obtain an estimate of the number of zeros for the function $F(\zeta(s+imh))$, where $\zeta(s)$ is the Riemann zeta-function, and $F: H(D) \to H(D)$ is a continuous function, $D = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1\}$.

Keywords: Riemann zeta-function, universality.

The distribution of zeros of zeta and L-functions is the central problem of analytic number theory, and the results in the field allow to solve many other important problems. For example, the location of non-trivial zeros of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, has a direct relation to the distribution of prime numbers. The best result in this direction asserts that $\zeta(s) \neq 0$ in the region

$$\sigma > 1 - \frac{c}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}, \quad |t| \geqslant t_0 > 0,$$

where $c > 0$ is an absolute constant. We remind that the Riemann hypothesis says that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$, thus by this hypothesis, $\zeta(s) \neq 0$ in the half-plane $\sigma > \frac{1}{2}$.

There are the zeta-functions for which the Riemann hypothesis is not true. For example, this holds for the Hurwitz-function $\zeta(s, \alpha)$, $0 < \alpha \leqslant 1$, defined, for $\sigma > 1$, by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. If α is *a* trancendental number, than [2] $\zeta(s, \alpha)$ has zeros in the strip $\frac{1}{2} < \sigma < 1$. Also, the derivative $\zeta'(s)$ has zeros in the strip $0 < \sigma < 1$.

For the investigation of zero-distribution of zeta-functions, universality theorems can be applied. The first universality theorem for the Riemann zeta-function has been proved by S.M. Voronin in [5]. The last version of this theorem is the following:

Theorem 1. *Suppose that K is a compact subset of the strip $D = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1\}$ with connected complement, and $f(s)$ is a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T]\colon \sup_{s\in K} \big|\zeta(s+i\tau) - f(s)\big| < \varepsilon\Big\} > 0.$$

Here meas$\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The proof of Theorem 1 is given, for example, in [1].

Also, a discrete version of Theorem 1 is known. Let $h > 0$ be a fixed number.

Theorem 2. *Suppose that K and $f(s)$ satisfy the hypotheses of Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \sharp\Big\{0 \leqslant m \leqslant N\colon \sup_{s\in K} \big|\zeta(s+imh) - f(s)\big| < \varepsilon\Big\} > 0.$$

In [3], certain discrete universality theorems were obtained for the composite function $F(\zeta(s))$.

We recall some of them. Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacte, and set

$$S = \big\{g \in H(D)\colon g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0\big\}.$$

Theorem 3. *Suppose that the number $\exp\{\frac{2\pi k}{h}\}$ is irrational for all $k \in \mathbb{Z} \setminus \{0\}$, and that $F : H(D) \to H(D)$ is a continuous function such that, for every open set $G \subset H(D)$, the set $(F^{-1}G \cap S)$ is non-empty. Let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \sharp\Big\{0 \leqslant m \leqslant N\colon \sup_{s\in K} \big|F\big(\zeta(s+imh)\big) - f(s)\big| < \varepsilon\Big\} > 0.$$

The next theorem is a simplification of Theorem 3.

Theorem 4. *Suppose that the number h, the set K and the function $f(s)$ satisfy the hypotheses of Theorem 3, and that $F : H(D) \to H(D)$ is a continuous function such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Then the assertion of Theorem 3 is true.*

Now let V be an arbitrary positive number. Define

$$D_V = \Big\{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1,\ |t| < V\Big\}$$

and

$$S_V = \big\{g \in H(D_V)\colon g^{-1}(s) \in H(D_V) \text{ or } g(s) \equiv 0\big\}.$$

Theorem 5. *Suppose that the number h, the set K and the function $f(s)$ satisfy the hypotheses of Theorem 3, and that $V > 0$ is such that $K \subset D_V$. Let $F : H(D_V) \to H(D_V)$ be a continuous function such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty. Then the assertion of Theorem 3 is true.*

We note that, differently from Theorem 2, the approximated function in Theorems 3–5 is not necessarily non-vanishing.

The aim of his note is to prove the following statement.

Theorem 6. *Suppose that the number* $\exp\{\frac{2\pi k}{h}\}$ *is irrational for all* $k \in \mathbb{Z}\setminus\{0\}$, *and that the function* F *is as in one of Theorems* 3–5. *Then, for arbitrary* σ_1 *and* σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, *there exists a constant* $c = c(\sigma_1, \sigma_2) > 0$ *such that the function* $F(\zeta(s + imh))$ *has a zero in the disc*

$$|s - \hat{\sigma}| \leqslant \frac{\sigma_2 - \sigma_1}{2}, \qquad \hat{\sigma} = \frac{\sigma_1 + \sigma_2}{2},$$

for more than cN *numbers* m, $0 \leqslant m \leqslant N$.

First we will remind the Rouché theorem.

Lemma 1. *Suppose that* G *is a region on the complex plane bounded by a closed continuous contour* L. *Let* $f_1(s)$ *and* $f_2(s)$ *be two analytic functions on* G, *and* $f_1(s) \neq 0$ *and* $|f_2(s)| < |f_1(s)|$ *on* L. *Then the functions* $f_1(s)$ *and* $f_1(s) + f_2(s)$ *have the same number of zeros on* G.

Proof of the lemma can be found, for example, in [4].

Proof of Theorem 6. Let

$$\sigma_0 = \max\left(\left|\sigma_1 - \frac{3}{4}\right|, \left|\sigma_2 - \frac{3}{4}\right|\right),$$

$f(s) = s - \hat{\sigma}$ and $0 < \varepsilon < \frac{\sigma_2 - \sigma_1}{20}$. Then, in virtue of Theorems 3–5, there exists a constant $c = c(\sigma_1, \sigma_2) > 0$ such that, for sufficently large N,

$$\frac{1}{N+1}\sharp\left\{0 \leqslant m \leqslant N\colon \sup_{|s-\frac{3}{4}|\leqslant\sigma_0} \bigl|F\bigl(\zeta(s + imh)\bigr) - f(s)\bigr| < \varepsilon\right\} > c. \tag{1}$$

The circle $|s - \hat{\sigma}| = \frac{\sigma_2 - \sigma_1}{2}$ lies in the disc

$$\left|s - \frac{3}{4}\right| \leqslant \sigma_0.$$

Therefore, for m satisfying (1), we have that

$$\max_{|s-\hat{\sigma}|=\frac{\sigma_2-\sigma_1}{2}} \bigl|F\bigl(\zeta(s + imh)\bigr) - (s - \hat{\sigma})\bigr| < \frac{\sigma_2 - \sigma_1}{20}.$$

This shows that the functions $(s - \hat{\sigma})$ and

$$F\bigl(\zeta(s + imh)\bigr) - (s - \hat{\sigma})$$

satisfy the hypotheses of the Rouché theorem in the disc $|s - \hat{\sigma}| \leqslant \frac{\sigma_2 - \sigma_1}{2}$. However, the function $s - \hat{\sigma}$ has precisily one zero $s = \hat{\sigma}$ in that disc. Therefore, by the Rouché theorem, the function $F(\zeta(s + imh))$ also has one zero in the disc $|s - \hat{\sigma}| \leqslant \frac{\sigma_2 - \sigma_1}{2}$. Since, in view of (1) the number of such m, $0 \leqslant m \leqslant N$, is larger that cN, this proves the theorem. □

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REZIUMĖ

Apie kai kurių sudėtinių funkcijų nulius
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Tarkime, kad $\zeta(s)$, $s = \sigma + it$, yra Rymano dzeta funkcija, $H(D)$, $D = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1\}$, yra analizinių funkcijų srityje D erdvė, o $F : H(D) \to (D)$ yra tolydi funkcija. Straipsnyje gautas funkcijos $F(\zeta(s + imh))$ nulių skaičiaus įvertis.

Raktiniai žodžiai: Rymano dzeta funkcija, universalumas.