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Solution of two-dimensional parabolic equation with nonlocal integral boundary conditions*

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Abstract. Two-dimensional parabolic equation with nonlocal integral boundary conditions in a rectangle domain in this paper is solved by alternating direction method. To find the solutions of this problem we are looking to solve a linear system of equations. This algorithm is realized on particular example and assess the error of solution.

Keywords: two-dimensional parabolic equation, nonlocal integral condition, finite-difference method, alternating direction method.

Introduction

We consider boundary value problem of two-dimensional parabolic equation in rectangular with two integral boundary conditions. We investigate boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad x, y \in \Omega = \{0 \leqslant x, y \leqslant 1\},\ 0 < t \leqslant T, \tag{1}$$

$$u(x,y,0) = \varphi(x,y), \quad x, y \in \Omega, \tag{2}$$

$$u(0,y,t) = \mu_1(y,t), \quad y \in \Omega, \quad 0 < t \leqslant T, \tag{3}$$

$$u(1,y,t) = \mu_2(y,t), \quad y \in \Omega, \quad 0 < t \leqslant T, \tag{4}$$

$$u(x,0,t) = \iint\limits_{\Omega} \gamma_3(x,\xi) u(\xi,\eta,t)\, d\xi\, d\eta + \mu_3(x,t), \quad x \in \Omega,\ 0 < t \leqslant T, \tag{5}$$

$$u(x,1,t) = \iint\limits_{\Omega} \gamma_4(x,\xi) u(\xi,\eta,t)]\, d\xi\, d\eta + \mu_4(x,t), \quad x \in \Omega,\ 0 < t \leqslant T. \tag{6}$$

Numerical methods for two-dimensional and three-dimensional parabolic equations with nonlocal conditions are considered in many articles (for example [1, 2, 3, 4, 5, 6]). The specifity of the problem (1)–(6) is that boundary conditions (5)–(6) include integral through entire domain Ω. Integral condition of this type in more general form is

$$u(x,y,t) = \iint\limits_{\Omega} K(x,y,\xi,\eta) u(\xi,\eta,t)\, d\xi\, d\eta + \mu(x,y,t), \tag{7}$$

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when $(x, y) \in \partial\Omega$. Parabolic equation with this type of condition is solved in article [7] by finite difference method when the kernel $K(x, y, \xi, \eta)$ satisfies

$$\iint\limits_{\Omega} K(x, y, \xi, \eta)\, d\xi\, d\eta \leqslant \rho < 1, \quad x, y \in \partial\Omega. \tag{8}$$

In our article assumption (8) is disposed. However, numerical experiment performed by our method confirms, that $K(x, y, \xi, \eta)$ should satisfy certain condition the difference scheme would be stable.

1 Numerical method

We apply alternating direction method for differential problem (1)–(6). So we have to solve two systems of difference equations. First we consider one-dimensional problem with boundary conditions:

$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^{n}}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+\frac{1}{2}} + \Lambda_2 u_{ij}^{n} + f_{ij}^{n+\frac{1}{2}}, \quad i, j = \overline{1, N-1}, \tag{9}$$

$$u_{0j}^{n+\frac{1}{2}} = \mu_{1_j}^{n+\frac{1}{2}}, \qquad u_{Nj}^{n+\frac{1}{2}} = \mu_{2_j}^{n+\frac{1}{2}}, \quad j = \overline{1, N-1}, \tag{10}$$

where

$$\Lambda_1 u_{ij}^{n+1} = \frac{u_{i-1,j}^{n+1} - 2u_{ij}^{n+1} + u_{i+1,j}^{n+1}}{h^2}, \quad i, j = \overline{1, N-1}, \tag{11}$$

$$\Lambda_2 u_{ij}^{n+1} = \frac{u_{i,j-1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j+1}^{n+1}}{h^2}, \quad i, j = \overline{1, N-1}. \tag{12}$$

We solve it by using Thomas algorithm and find solution $(n + \frac{1}{2})$-th layer of time. Then we are looking for solution in $(n + 1)$-th layer of time from second difference problem with nonlocal boundary conditions

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+\frac{1}{2}} + \Lambda_2 u_{ij}^{n+1} + f_{ij}^{n+1}, \quad i, j = \overline{1, N-1}, \tag{13}$$

$$u_{i0}^{n+1} = h^2 \sum_{k=1}^{N-1} \sum_{l=0}^{N} \gamma_{3_{ik}} \rho_{kl} u_{kl}^{n+1} + g_{1_{i0}}^{n+1} + \mu_{3_i}^{n+1}, \quad i = \overline{1, N-1}, \tag{14}$$

$$u_{iN}^{n+1} = h^2 \sum_{k=1}^{N-1} \sum_{l=0}^{N} \gamma_{4_{ik}} \rho_{kl} u_{kl}^{n+1} + g_{2_{iN}}^{n+1} + \mu_{4_i}^{n+1}, \quad i = \overline{1, N-1}, \tag{15}$$

where $g_{1,i0}^{n+1}$ and $g_{2,iN}^{n+1}$ depends on u_{0l}^{n+1}, u_{Nl}^{n+1}, $l = \overline{0, N}$, $h = \frac{1}{N}$, $\tau = \frac{1}{M}$, and

$$\rho_{ij} = \begin{cases} 1, & j \neq 0, N, \\ 1/2, & j = 0,\ j = N. \end{cases}$$

The main question is how to solve system of difference equations (13)–(15).

Let's rewrite equation (13) in the shorter way and have

$$au_{ij-1}^{n+1} - cu_{ij}^{n+1} + bu_{ij+1}^{n+1} = F_{ij}^{n+1}, \tag{16}$$

where $a = \frac{\tau}{2h^2}$, $b = \frac{\tau}{2h^2}$, $c = 1 + \frac{\tau}{h^2}$ and $c > a + b$.

Solution of the equation (16) is taken as following

$$u_{ij}^{n+1} = c_1^{n+1} {u_{ij}^{(1)}}^{n+1} + c_2^{n+1} {u_{ij}^{(2)}}^{n+1} + {u_{ij}^{(0)}}^{n+1}, \quad j = 0, 1, 2, \dots, N, \tag{17}$$

where $u_{ij}^{(1)}$ and $u_{ij}^{(2)}$ are two solutions of homogeneous system (13) with boundary conditions $u_{i0}^{(1)} = 1$, $u_{iN}^{(1)} = 0$, and $u_{i0}^{(2)} = 0$, $u_{iN}^{(2)} = 1$, and $u_{ij}^{(0)}$ is solution of the system (13) with $u_{i0}^{(0)} = u_{iN}^{(0)} = 0$.

Remark 1. c_1^{n+1} and c_2^{n+1} should be choosen in the way boundary conditions (14) and (15) would be correct.

So $c_1^{n+1} \equiv u_{i0}^{n+1}$ and $c_2^{n+1} \equiv u_{iN}^{n+1}$.

We again rewrite expresion (17) as

$$u_{ij}^{n+1} = u_{i0}^{n+1} u_j^{(1)} + u_{iN}^{n+1} u_j^{(2)} + {u_{ij}^{(0)}}^{n+1}. \tag{18}$$

When we put (18) to conditions (14) and (15) recieving $2(N-1)$ linear algebraic equations

$$\begin{aligned} u_{i0}^{n+1} =& h \sum_{k=1}^{N-1} \gamma_{3_{ik}} u_{k0}^{n+1} * h \sum_{l=0}^{N} \rho_l u_l^{(1)} + h \sum_{k=1}^{N-1} \gamma_{3_{ik}} u_{kN}^{n+1} * h \sum_{l=0}^{N} \rho_l u_l^{(2)} \\ &+ h^2 \sum_{k=1}^{N-1} \sum_{l=0}^{N} \gamma_{3_{ik}} {u_{kl}^{(0)}}^{n+1} + {g_i^{(1)}}^{n+1}, \end{aligned} \tag{19}$$

$$\begin{aligned} u_{iN}^{n+1} =& h \sum_{k=1}^{N-1} \gamma_{4_{ik}} u_{k0}^{n+1} * h \sum_{l=0}^{N} \rho_l u_l^{(1)} + h \sum_{k=1}^{N-1} \gamma_{4_{ik}} u_{kN}^{n+1} * h \sum_{l=0}^{N} \rho_l u_l^{(2)} \\ &+ h^2 \sum_{k=1}^{N-1} \sum_{l=0}^{N} \gamma_{4_{ik}} {u_{kl}^{(0)}}^{n+1} + {g_i^{(1)}}^{n+1}, \end{aligned} \tag{20}$$

with $2N-2$ unknowns u_{k0}^{n+1}, u_{kN}^{n+1}, $k = \overline{1, N-1}$. We rewrite equations (19) and (20) in the form of matrix equation:

$$Au = F, \tag{21}$$

where matrix A is

$$A = \begin{pmatrix} (1 - \gamma_{3_{1,1}}\alpha) & \dots & -\gamma_{3_{1,N-1}}\alpha & -\gamma_{3_{1,1}}\beta & \dots & -\gamma_{3_{1,N-1}}\beta \\ -\gamma_{3_{2,1}}\alpha & \dots & -\gamma_{3_{2,N-1}}\alpha & -\gamma_{3_{2,1}}\beta & \dots & -\gamma_{3_{2,N-1}}\beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\gamma_{3_{N-1,1}}\alpha & \dots & (1 - \gamma_{3_{N-1,N-1}}\alpha) & -\gamma_{3_{N-1,1}}\beta & \dots & -\gamma_{3_{N-1,N-1}}\beta \\ -\gamma_{4_{1,1}}\alpha & \dots & -\gamma_{4_{1,N-1}}\alpha & (1 - \gamma_{4_{1,1}}\beta) & \dots & -\gamma_{4_{1,N-1}}\beta \\ -\gamma_{4_{2,1}}\alpha & \dots & -\gamma_{4_{2,N-1}}\alpha & -\gamma_{4_{2,1}}\beta & \dots & -\gamma_{4_{2,N-1}}\beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\gamma_{4_{N-1,1}}\alpha & \dots & -\gamma_{4_{N-1,N-1}}\alpha & -\gamma_{4_{N-1,1}}\beta & \dots & (1 - \gamma_{4_{N-1,N-1}}\beta) \end{pmatrix}.$$

Here

$$\alpha = h^2 \sum_{l=0}^{N} \rho_l u_l^{(1)}, \qquad \beta = h^2 \sum_{l=0}^{N} \rho_l u_l^{(2)}. \tag{22}$$

We apply Gausian elimimation method for solving the system of equations (21) and obtain solutions u_{k0}^{n+1} and u_{kN}^{n+1} ($k = \overline{1, N-1}$). The solutions u_{k0}^{n+1} and u_{kn}^{n+1} ($k = \overline{1, N-1}$) we put into equation (18) and then find solution we were looking in $(n+1)$-th layer of time.

2 Numerical experiment

The method considered in this paper for the solution of the system of difference equations is applied for solving of testing example (1)–(6) with $\gamma_3(x,\xi) = \gamma_3(x)$ and $\gamma_4(x,\xi) = \gamma_4(x)$. Choosing appropriate functions $f(x,y,t)$, $\varphi(x,y)$, $\mu_1(y,t)$, $\mu_2(y,t)$, $\mu_3(x,t)$, $\mu_4(x,t)$, $\gamma_3(x)$ and $\gamma_4(x)$ the solutions would be $u(x,y,t) = \sin(\pi x)\sin(\pi y)e^{2t}$. The accuracy of solution depends on the selection of functions $\gamma_3(x)$ and $\gamma_4(x)$. Errors of solution u_{ij}^{n+1} ($i,j = \overline{1, N-1}$) are given in four tables, with functions $\gamma_3(x)$ and $\gamma_4(x)$.

Here

$$\varepsilon_u = \max_{0 \leqslant i,j \leqslant N} |z_{ij}| = \max_{0 \leqslant i,j \leqslant N} \left| u(x_i, y_j, t^n) - u_{ij}^n \right|. \tag{23}$$

The results of numerical experiment shows, that difference scheme is stable (Table 1), when

$$|\gamma_3| < 1, \qquad |\gamma_4| < 1. \tag{24}$$

The similar conclusion was derived in some other papers, where more simple non-local conditions were considered [5, 7]. However our difference scheme is stable even in the case, when γ_3 and γ_4 don't satisfy conditions (24) but at the same time are negative (Table 2).

Furthermore, taken with some positive values of γ_3 and γ_4, not satisfy conditions (24) the errors of the solutions in the case of $T = 1$, also are of the order

Table 1.

h	:	0.2	0.1	0.05	0.025
τ	:	0.1	0.025	0.00625	0.0015625
		ε_u			
$\gamma_3(x) = 0$, $\gamma_4(x) = 0$	:	0.5476	0.1485	0.0369	0.0092
$\gamma_3(x) = 1$, $\gamma_4(x) = 1$	:	0.5762	0.1567	0.0391	0.0098
$\gamma_3(x) = e^x$, $\gamma_4(x) = 0.1e^x$	:	0.5809	0.1556	0.0388	0.0097

Table 2.

h	:	0.2	0.1	0.05	0.025
τ	:	0.1	0.025	0.00625	0.0015625
		ε_u			
$\gamma_3(x) = -2e^x$, $\gamma_4(x) = -10e^x$	:	0.5271	0.1408	0.0350	0.0087
$\gamma_3(x) = -5$, $\gamma_4(x) = -10$	:	0.5245	0.1412	0.0351	0.0088

Table 3.

h	:	0.2	0.1	0.05	0.025
τ	:	0.1	0.025	0.00625	0.0015625
		ε_u			
$\gamma_3(x) = 5$, $\gamma_4(x) = -0.3$	:	317.393	6.224	1.065	0.236
$\gamma_3(x) = 2.3$, $\gamma_4(x) = 2.3$	:	36.578	1.9426	0.3964	0.0940

Table 4.

h	:	0.2	0.1	0.05	0.025
τ	:	0.1	0.025	0.00625	0.0015625
		ε_u			
$\gamma_3(x) = 5$, $\gamma_4(x) = 1$	:	1.5035	4.531×10^6	2.895×10^5	6.736×10^4
$\gamma_3(x) = 2e^x$, $\gamma_4(x) = e^x$	:	1.229×10^3	18.754	16.950	5.419

$O(\tau + h^2)$ (Table 3). But with the growth of T, the scheme generally becomes unstable. Moreover, in the case of sufficiently big meanings of γ_3 and γ_4 the instability of the difference scheme could be seen even, then $T = 1$ (Table 4).

3 Conclusions

The numerical experiment based on the method provided in this paper was performed with various meanings of γ_3 and γ_4 investigating the influence of these parameters on the stability of the difference scheme. It was proven in the papers [8] that stability of the difference scheme depends on the structure of the spectrum of matrix of this scheme. It was shown, that in the case of more simple nonlocal conditions (see [5]) the difference scheme might be stable with considerably big in absolute values negative meanings of γ_3 and γ_4. In our case, considering the results of the numerical experiment, this characteristic was observed. More precised conclusions about the stability of the difference scheme could be obtained investigating the spectrum of the system of difference equations.

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REZIUMĖ

Dvimatės parabolinės lygties su dviem nelokaliosiomis integralinėmis sąlygomis sprendimas

K. Jakubėlienė

Straipsnyje išnagrinėtas dvimatės parabolinės lygties su nelokaliosiomis integralinėmis kraštinėmis sąlygomis sprendimas kintamųjų krypčių metodu. Uždavinio sprendinį randame išsprendę tiesinę lygčių sistemą. Pateikti skaitinio eksperimento rezultatai.

Raktiniai žodžiai: dvimatė parabolinė lygtis, nelokalioji integralinė sąlyga, baigtinių skirtumų metodas, kintamųjų krypčių metodas.