Lietuvos matematikos rinkinys
Proc. of the Lithuanian Mathematical Society, Ser. A
Vol. **53**, 2012, 30–35

ISSN 0132-2818
www.mii.lt/LMR/

The use of Lambert function method for analysis of a multidimensional control system with delays and with structure of the complete graph

Irma Ivanovienė, Jonas Rimas

Department of Applied Mathematics, Kaunas University of Technology
Studentų 50, LT-51368, Kaunas
E-mail: irma.ivanoviene@yahoo.com, jonas.rimas@ktu.lt

Abstract. The mathematical model of the mutual synchronization system with structure having form of the complete graph and composed of n ($n \in N$) oscillators, is investigated. The mathematical model of the system is the matrix differential equation with delayed argument. The step responses matrix of the system is obtained applying the Lambert function method. The transition processes are investigated using obtained step responses of the system.

Keywords: synchronization system, differential equations, delayed arguments, Lambert function.

Introduction

Control systems are used in different areas of human activity. Manufacture, transport, transmission and distribution of information are only few of these areas. Usually control systems are being investigated applying their mathematical models. The requirements to these models are growing with the requirements of the accuracy of these investigations . Therefore, the delays of the transferred signals along the control system, often must be included into the mathematical model. Presence of delays at mathematical model of a control system makes its research more difficult. Though there are great achievements in the investigation of control systems with delays, the works of analytical investigation of such systems are required.

1 Formulation of the problem

In the presented work the dynamics of the multidimensional control system with delays and with structure of the complete graph is investigated. The mathematical model of this system is the matrix differential equation with delayed argument [1, 2, 3, 4]

$$x'(t) + B_1 x(t) + B_2 x(t-\tau) = z(t), \quad x(t) = \phi(t),\ t \in [-\tau, 0], \tag{1}$$

where $x(t) = (x_1(t)x_2(t)\cdots x_n(t))^T$ is the desired vector function, T denotes the operation of transposition, τ is a constant time delay, $\phi(t)$ is a vector valued initial function, $z(t)$ is a free term (continuous function depending on the initial conditions),

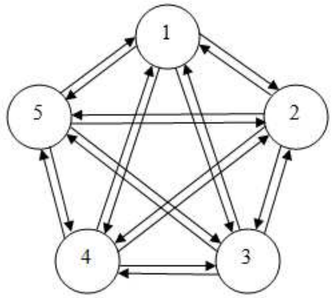


Fig. 1. The scheme of internal links of the system, when $n = 5$.

κ is a coefficient, B_1 and B_2 are $n \times n$ ($n \in N$) numerical matrices ($B_1, B_2 \in R^{n\times n}$),

$$B_1 = \kappa E, \qquad B_2 = \frac{\kappa}{n-1} B, \qquad \{B\}_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j. \end{cases} \tag{2}$$

($E \in R^{n\times n}$ is the identity matrix, matrix $B \in R^{n\times n}$ outlines the structure of the internal links of the system).

As an example of a control system, described by equation (1), the mutual synchronization system of the communication network having structure of the complete graph and composed of n oscillators can be pointed out [2] (in the Fig. 1 the scheme of the internal links of the system composed of 5 oscillators is presented). In this case the symbol $x_i(t)$ in (1) stands for the phase of the i-th oscillator. Taking into account the system's reaction to the unit jumps of the phases of oscillations of the oscillators, we shall investigate the transient processes in the synchronization system. For this purpose, firstly, we shall find the step responses matrix of the synchronization system.

2 Step responses matrix of the system

The matrix $h(t) = h_{ij}(t)$ we shall call the step responses matrix of the synchronization system. The entry $h_{ij}(t)$ $(i, j = \overline{1,n})$ of this matrix is the response of the i-th oscillator oscillation phase to a unit jump in the j-th oscillator oscillation phase. We shall find the matrix $h(t)$.

When the increment of the phase of the j-th oscillator takes form of the unit jump the increment of the free term of equation (1) can be expressed as follows

$$\Delta z(t) = \delta(t) I^{(j)}, \tag{3}$$

here $I^{(j)}$ is the matrix-column all elements of which are zeros except the j-th element, which is equal to 1, $\delta(t)$ is the Dirac delta-function. Taking this into account and using (1), we get the following differential equation for step responses $h_{ij}(t)$ $(i, j = \overline{1,n})$ of the system:

$$\begin{gathered} \big(h_j(t)\big)' + B_1 h_j(t) + B_2 h_j(t-\tau) = \delta(t) I^{(j)}, \quad j = \overline{1,n}, \\ h_j(t) = 0, \quad t \in [-\tau, 0], \end{gathered} \tag{4}$$

here $h_j(t) = (h_{1j}(t)h_{2j}(t)\dots h_{nj}(t))^T$ is the j-th column of the step responses matrix $h(t)$, matrices B_1 and B_2 are defined by (2), respectively.

Using the solution, found on the interval $[0,\tau]$, the differential equation (2) on the interval $[\tau,+\infty)$ can be presented as homogeneous matrix delay differential equation:

$$\begin{aligned}\big(h_j(t)\big)' + B_1 h_j(t) + B_2 h_j(t-\tau) = 0, \quad j = \overline{1,n},\\ h_j(t) = \big(h_{1j}(t)h_{2j}(t)\dots h_{nj}(t)\big)^T = \phi_j(t), \quad t\in[0,\tau],\end{aligned} \tag{5}$$

here $\phi_j(t)$ is the preshape (initial) vector-function. The entries of the vector-function $\phi_j(t)$ assume the following values:

$$\phi_{ij}(t) = \big(\phi_j(t)\big)_i = \begin{cases} e^{-\kappa t}1(t), & \text{if } i = j,\\ 0, & \text{if } i \neq j,\end{cases} \tag{6}$$

here $1(t)$ is the Heaviside step function.

Applying the Lambert function method, the solution of (2) on the interval $[\tau,+\infty)$ can be expressed as follows [4]:

$$h_j(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k(j) = \lim_{N\to\infty} \sum_{k=-N}^{N} e^{S_k t} C_k(j), \quad j=\overline{1,n},\ t\in[\tau,+\infty), \tag{7}$$

here

$$S_k = \frac{1}{\tau} W_k\big(-B_2\tau e^{B_1\tau}\big) - B_1,$$

$W_k(-B_2\tau e^{B_1\tau})$ is the value of the k-th branch $W_k(H)$ of the matrix Lambert function $W(H)$ at $H = -B_2\tau e^{B_1\tau}$, $C_k(j)$ are the complex valued vectors corresponding to the preshape vector function $\phi_j(t)$ (see (2) and (6)). From (6) follows the approximate expression for $h_j(t)$:

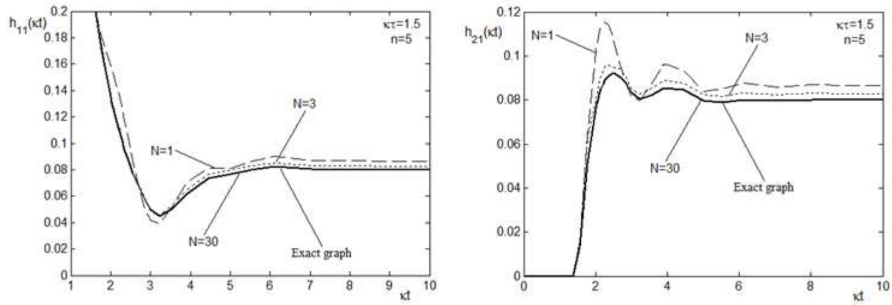
$$h_j(t) = \sum_{k=-N}^{N} e^{S_k t} C_k(j), \quad j=\overline{1,n},\ t\in[\tau,+\infty), \tag{8}$$

here N is a sufficiently large natural number.

3 Comparing the Lambert function method with the method of consequent integration

The solution of homogeneous matrix delay differential equation (2) is presented by the infinite functional series (see (7)), which determines the exact solution. In the real calculations we apply the approximate formulas (8), obtained from (7) with finite N ($2N+1$ indicates the number of branches of the Lambert W function, which are used in calculations of the solution).

We shall investigate the rate of convergence of the approximate solution to the exact solution with increasing N. For this purpose we shall apply the exact expression of the step response of the mutual synchronization system with structure having form of the complete graph. This expression we shall find by the method of consequent integration (method of “steps”) [3].



(a) (b)

Fig. 2. Graphs of the step responses $h_{i1}(\kappa t)$ at different values of N.

The solution of (1), applying the Laplace transform, we present as follows [2]:

$$x(t) \div \sum_{l=1}^{L} \left(A^{-1}B_2e^{-pt}\right)^l A^{-1}Z(p), \quad 0 \leqslant t \leqslant (L+1)\tau,$$

here $A = pE - B_1 = (p+\kappa)E$, $Z(p) \div z(t)$, $Z(p)$ is the Laplace transform of the vector function $z(t)$ (sign $\div$ links function with its Laplace transform), $L = 0, 1, 2, \ldots$. Taking into account (2), we write

$$x(t) \div \sum_{l=0}^{L} \left(\frac{\kappa}{n-1}\right)^l \frac{1}{(p+\kappa)^{l+1}} e^{-pl\tau} B^l Z(p), \quad 0 \leqslant t \leqslant (L+1)\tau.$$

Write down the step responses matrix of the system. Using (6), we obtain

$$h(t) = \left(h_{ij}(t)\right) \div \sum_{l=0}^{L} \left(\frac{\kappa}{n-1}\right)^l \frac{1}{(p+\kappa)^{l+1}} e^{-pl\tau} B^l, \quad 0 \leqslant t \leqslant (L+1)\tau.$$

The inverse Laplace transform, applied to the right hand side of the latter expression, gives

$$h_{ij}(t) = \sum_{l=0}^{L} \left(\frac{\kappa}{n-1}\right)^l \left\{B^l\right\}_{ij} \frac{(t-l\tau)^l}{l!} e^{-\kappa(t-l\tau)} 1(t-l\tau), \quad 0 \leqslant t \leqslant (L+1)\tau,$$

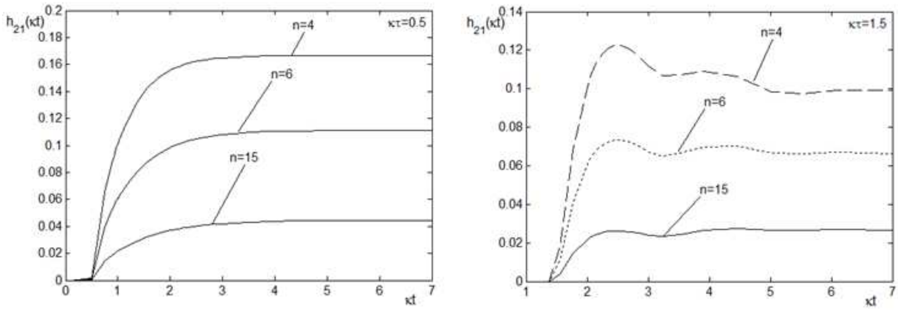
here $\{B^l\}_{ij}$ is the ij-th element of the matrix B^l.

The step response of mutual synchronization system with a structure having form of the complete graph, computed by the method of consequent integration (the exact method) and by Lambert W function method with different values of N, are presented in the Fig. 2. As we see from this figure, increasing N the approximate solution approaches the exact solution.

The maximal relative errors $\delta_{\max}$ obtained for $\kappa t \in [0, +\infty)$ using different values of N are presented in the Table 1, when $n = 5$ and $n = 15$. As we see from the table for $N = 50$ the maximal relative error is not greater than 0.06 (with increase of N and with decrease of n the maximum relative error has tendency to decrease). Such accuracy is sufficient for practical applications.

Table 1.

	$n=5$				$n=15$			
N	1	3	30	50	1	3	30	50
$\delta_{\max}$	0.3480	0.2359	0.0567	0.0351	0.3983	0.2962	0.1825	0.0522



(a) (b)

Fig. 3. Graphs of the step responses $h_{21}(\kappa t)$.

4 Results of calculations

The transients in the synchronization system were investigated applying derived formulas. Some results of calculations are presented in Fig. 3 as graphs of step responses.

For the calculation of the step responses we have applied the approximate formula (8) with $N = 50$ (this means that we have used 101 branches of the Lambert W function in the computations). With such N the relative error is not greater than 0.06 for any κt on the base of the 3-th section. So the graphs of the step responses, presented below, are sufficiently accurate (in the presented figures these graphs practically coincide with the exact ones).

In Fig. 3 the graphs of the step responses $h_{ij}(\kappa t)$ are given for different values of product $\kappa\tau$ and for different numbers of oscillators in the synchronization system. From the figure we see that the duration of transients in the synchronization system depends on the magnitude of the product $\kappa\tau$ With increase of $\kappa\tau$ the duration of transients in the system tends to increase. With increase of n the duration of transients in the system changes marginally. The transients get oscillatory features if $\kappa\tau \geqslant 1.5$.

5 Conclusions

1. The Lambert W function method is used for computing step responses for the synchronization system. It is shown that using 101 branches of the Lambert W function (taking $N = 50$) in calculations of step responses $h_{ij}(\kappa t)$ the relative error is not greater than 0.06 for any κt.
2. The Lambert W function method has the advantage in comparison with a method of consequent integration (method of "steps"), as time of calculation of the step response by this method does not depend on delay size, whereas time of

calculation of the step response by means of a method of consequent integration is in inverse proportion to the delay size.

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REZIUMĖ

Daugiamatės valdymo sistemos su vėlavimais, kurios struktūra pilnas grafas, tyrimas

I. Ivanovienė, J. Rimas

Analizuotas tarpusavio sinchronizacijos sistemos, sudarytos iš n generatorių ir turinčios pilnojo grafo struktūrą, matematinis modelis. Sistemos matematinis modelis yra matricinė diferencialinė lygtis su vėluojančiu argumentu. Sistemos pereinamosios funkcijos gautos taikant Lamberto funkcijų metodą. Naudojant gautas pereinamąsias funkcijas tirti sistemos pereinamieji procesai.

Raktiniai žodžiai: sinchronizacijos sistema, diferencialinė lygtis, vėluojantis argumentas, Lamberto funkcija.