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Explicit difference schemes for a pseudoparabolic equation with an integral condition

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Abstract. The aim of this paper is to analyze three layer explicit schemes for a pseudoparabolic equation with different boundary conditions, including nonlocal ones. The numerical results are presented.

Keywords: pseudoparabolic equation, three layer explicit scheme, eigenvalue problem.

1 Statement of the problem

Much attention in mathematical literature has been paid to the third order pseudoparabolic equation with nonlocal conditions during the last two decades. Both theoretical issues [1, 5] and numerical solution analysis methods [2, 4] are considered. When solving this equation with nonlocal or classical boundary conditions by the finite difference method, we can notice an important property: it is impossible to write two-layer explicit difference schemes to this equation. The stability and convergence of implicit difference schemes for a pseudoparabolic equation of Dirichlet type with boundary-value conditions were considered in [3].

At the first, we consider a linear pseudoparabolic problem with the classical boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) + f(x,t), \quad t \geqslant 0,\ 0 < x < 1, \tag{1}$$

$$u(0,x) = \varphi(x), \qquad u(0,t) = \mu_1(t), \tag{2}$$

$$u(1,t) = \mu_2(t). \tag{3}$$

If the parameter $\eta \neq 0$, a third-order derivative, present in equation (1), can be approximated in the following way (Fig. 1):

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) \sim \frac{1}{\tau}\left(\frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2} - \frac{u_{i-1}^{n} - 2u_i^{n} + u_{i+1}^{n}}{h^2}\right). \tag{4}$$

When changing a derivative in a simple standard way, we face the following difficulties: because of different signs, obtained in approximation (4), it is impossible to prove the maximum principle.

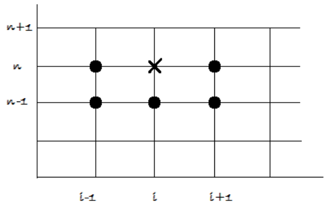

Fig. 1.

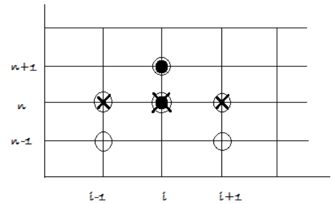

Fig. 2.

2 Three-layer explicit difference scheme (I) in the classical case

Since it is not clear how to write a two-layer explicit scheme for pseudoparabolic equation (1), we shall further consider three-layer explicit difference schemes.

Let us write an explicit scheme for equation (1) following the pattern, presented in (Fig. 2).

A difference scheme for equation (1) is

$$\frac{u_i^{n+1}-u_i^n}{\tau} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} + \eta\frac{\frac{u_{i-1}^n - u_{i-1}^{n-1}}{\tau} - 2\frac{u_i^{n+1}-u_i^n}{\tau} + \frac{u_{i+1}^n - u_{i+1}^{n-1}}{\tau}}{h^2} + f_i^{n+1}. \quad (5)$$

The approximation error of this scheme is $O(\tau + h^2 + \frac{\tau}{h^2})$. Thus, we have presented a three-layer difference scheme a typical feature of which is that the approximation error depends not only on the values τ and h^2, but also on the quantity $\frac{\tau}{h^2}$.

3 Stability of a three-layer explicit difference scheme

In the sequel, we deal with the stability of scheme (5). Let us write equation (5) as follows:

$$\begin{aligned}
&u_i^{n+1} - u_i^n \\
&\quad = \tau\Lambda u_i^n + \eta\frac{u_{i-1}^n - 2u_i^{n+1} + u_{i+1}^n}{h^2} - \eta\frac{u_{i-1}^{n-1} - 2u_i^n + u_{i+1}^{n-1}}{h^2} + \tau f \\
&\quad = \tau\Lambda u_i^n + \eta\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} + 2\eta\frac{u_i^n}{h^2} - 2\eta\frac{u_i^{n+1}}{h^2} \\
&\qquad - \eta\frac{u_{i-1}^{n-1} - 2u_i^{n-1} + u_{i+1}^{n-1}}{h^2} - 2\eta\frac{u_i^{n-1}}{h^2} + 2\eta\frac{u_i^n}{h^2} + \tau f \\
&\quad = \tau\Lambda u_i^n + \eta\Lambda u_i^n - \eta\Lambda u_i^{n-1} - 2\eta\frac{u_i^{n+1}}{h^2} - 2\eta\frac{u_i^{n-1}}{h^2} + 4\eta\frac{u_i^n}{h^2} + \tau f,
\end{aligned}$$

where $\Lambda u_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2}$.

After arranging we obtain

$$\left(1 + \frac{2\eta}{h^2}\right)u_i^{n+1} - \left(1 + \tau\Lambda + \eta\Lambda + \frac{4\eta}{h^2}\right)u_i^n + \left(\eta\Lambda + \frac{2\eta}{h^2}\right)u_i^{n-1} = \tau f. \quad (6)$$

or in a matrix form

$$Au^{n+1} + Bu^n + Cu^{n-1} = \tau f^n,$$

where

$$A = \left(1 + \frac{2\eta}{h^2}\right)E, \qquad B = -\left((1 + \frac{4\eta}{h^2})E + (\tau + \eta)\Lambda\right), \qquad C = \eta\Lambda + \frac{2\eta}{h^2}E.$$

Let us present this difference scheme as two-layer [7]. To this end, denote

$$v^n = \begin{pmatrix} u^n \\ u^{n-1} \end{pmatrix}, \qquad v^{n+1} = \begin{pmatrix} u^{n+1} \\ u^n \end{pmatrix}.$$

The vector v^{n+1} is of $2(N-1)$-order.

Using the vectors v^n and v^{n+1}, we can write the obtained matrix expression in shorter way:

$$v^{n+1} = Sv^n + g, \quad \text{where } S = \begin{pmatrix} -A^{-1}B & -A^{-1}C \\ E & 0 \end{pmatrix}. \tag{7}$$

To explore the stability of difference scheme (7) we use the following stability condition [6] $|\lambda(S)| < 1$.

Next we consider the problem of eigenvalues

$$Sv = \mu v, \tag{8}$$

which yields the equality $\det(S - \mu E) = 0$.

Let us expand the determinant of the matrix $S - \mu E$:

$$\begin{vmatrix} -A^{-1}B - \mu E & -A^{-1}C \\ E & -\mu E \end{vmatrix} = \left|\mu A^{-1}B + \mu^2 E + A^{-1}C\right| = 0.$$

So we can write expression $\det(S - \mu E) = 0$ as follows:

$$\det\left(\mu^2 A + \mu B + C\right) = 0.$$

Hence we find out that eigenvalue problem (8) is equivalent to the following nonlinear eigenvalue problem:

$$\mu^2 Av + \mu Bv + Cv = 0. \tag{9}$$

Thus, the main goal is to find the eigenvalues of matrix S. Therefore, in the sequel we will consider nonlinear eigenvalue problem (9). Let us consider an eigenvalue problem

$$\Lambda u + \lambda u = 0,$$

or in other form

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = \overline{1, N-1}, \qquad u_0 = u_N = 0.$$

Since matrices A, B and C have the same eigenvectors as matrix Λ, we have $\mu^2 Au + \mu Bu + Cu = 0$. Hence we derive $\mu^2\lambda_A u + \mu\lambda_B u + \lambda_C u = 0$ or

$$\mu^2\lambda_A + \mu\lambda_B + \lambda_C = 0. \tag{10}$$

If we know λ_A, λ_B, λ_C, then after solving this equation, we find the values of μ. Since we know from expressions A, B, C that

$$\lambda_A = 1 + \frac{2\eta}{h^2}, \qquad \lambda_B = -\biggl(1 + \frac{4\eta}{h^2} + (\tau+\eta)\lambda\biggr), \qquad \lambda_C = \eta\lambda + \frac{2\eta}{h^2},$$

we easily obtain from equation (10) that for $r \leqslant h^2/4$ and $0 < \lambda \leqslant 4/h^2$ the inequality is true

$$\max_i |\mu_i| \leqslant 1.$$

It means that the stability condition for a difference equation (7) is fulfilled.

Consequently, the explicit difference scheme for a pseudoparabolic equation is stable, no matter whether we solve the problem with the classical or nonlocal conditions.

4 Thee-layer explicit difference scheme (II) in the classical case

In this section, we shall analyze a little bit different explicit difference scheme, formed according to the pattern, presented in (Fig. 3), which has more points in the upper layer.

Let us present this scheme:

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} + \eta\frac{\frac{u_{i-1}^{n+1} - u_{i-1}^n}{\tau} - 2\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{u_{i+1}^n - u_{i+1}^{n-1}}{\tau}}{h^2} + f_i^n. \tag{11}$$

The approximation error of this scheme also is $O(\tau + h^2 + \frac{\tau}{h^2})$. In the general case, this scheme improves the stability.

Since it is complicated to study the stability of this explicit scheme theoretically, we have done several numerical calculation experiments. We have noticed that the solution error ε is directly connected with the quantity $\frac{\tau}{h^2}$ (see Table 1).

The results, given in Table 1, have shown that when reducing the steps τ, h and the ratio $\frac{\tau}{h^2}$, the error ε decreases.

Dependance of the error ε on T is illustrated in Fig. 4. Hence we see that, using low T values ($T = 1, 2, 4$), the error tends to zero, however, by increasing T we see

Table 1. Values of the error $\varepsilon = \max_{1\leqslant i\leqslant N-1} |u(x_i, t_j) - u_i^j|$, $T = 1$, $\eta = 0.1$.

h	τ	$\frac{\tau}{h^2}$	ε
$\frac{1}{20}$	$\frac{1}{800}$	0.5	0.0074
$\frac{1}{20}$	$\frac{1}{1600}$	0.25	0.0037
$\frac{1}{20}$	$\frac{1}{3200}$	0.125	0.0018

Calculation results are given in this table, where the exact solution of a differential problem is $u(x,t) = \sin(x)e^t$.

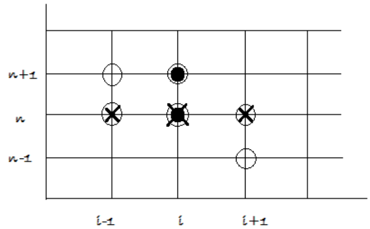

Fig. 3.

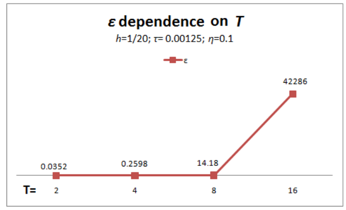

Fig. 4.

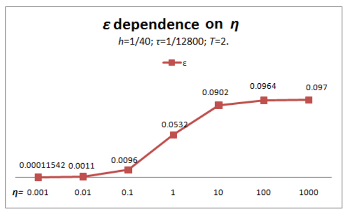

Fig. 5.

that the error is considerably growing, which means that the explicit difference scheme is not asymptotically stable.

The influence of the parameter η on the stability is not so susceptible. In Fig. 5, it is graphically displayed how the calculation error reacts to an increase of the parameter η.

5 Three-layer explicit difference scheme (II) with a nonlocal condition

We study problem (1)–(3), where instead of classical boundary condition (3) there is a nonlocal condition:

$$u(1,t) = \gamma \int_0^1 u(x,t)\,dx + \mu_2(t). \tag{12}$$

In this case, the eigenvalue problem of the operator Λ is defined as follows:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = \overline{1, N-1},$$
$$u_N^{n+1} = \gamma h \left(\frac{u_0 + u_N}{2} + \sum_{i=1}^{N-1} u_i \right), \qquad u_0 = 0.$$

If the parameter $\gamma > 2$, then both in the differential case and difference case there exist a negative eigenvalue λ.

After the calculations we have defined that, at low T values, it is possible to use explicit difference schemes for a pseudoparabolic problem with the classical conditions.

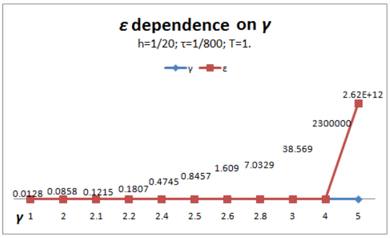

Fig. 6.

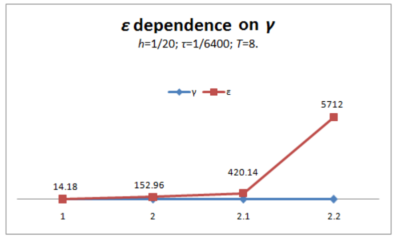

Fig. 7.

In the case of a problem with a nonlocal condition, at low values of the parameter γ, the explicit schemes act as stable. Instability displays itself at higher values of γ (see Fig. 6). If we take a larger T value ($T = 8$), instability manifests even sooner (see Fig. 7).

Consequently, when taking low T values, we can perform calculations using explicit schemes, however, at larger T values, we ought to calculate using implicit difference schemes.

References

[1] A. Bouziani. Solvability of nonlinear pseudoparabolic equation with nonlocal boundary condition. *Nonlin. Anal. Theor. Meth. Appl.*, **55**:883–904, 2003.

[2] A. Bouziani and N. Merazga. Solution to a semilinear pseudoparabolic problem with integral condition. *Elect. J. Diff. Equat.*, **115**:1–18, 2006.

[3] W.H. Ford and T.W. Ting. Stability and convergence of difference approximations to pseudoparabilic partial differential equations. *Mat. Comput.*, **27**(124):737–743, 1973.

[4] J. Jachimavičienė, Ž. Jesevičiūtė and M. Sapagovas. The stability of finite-difference schemes for a pseudoparabolic equation with nonlocal conditions. *Numer. Funct. Anal. Optimiz.*, **30**(9):988–1001, 2009.

[5] S. Mesloub. A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation. *J. Math. Anal. Appl.*, **316**:189–209, 2006.

[6] M. Sapagovas. On the stability of a finite difference scheme for nonlocal parabolic boundary-value problems. *Lith. Math. J.*, **48**(3):339–356, 2008.

[7] M. Sapagovas. On the spectral properties of three-layer difference schemes for parabolic equations with nonlocal conditions. *Diff. Equat.*, **48**(7):1033–1041, 2012.

REZIUMĖ

Išreikštinės skirtuminės schemos pseudoparabolinei lygčiai su integraline sąlyga

J. Jachimavičienė

Šio straipsnio tikslas – išnagrinėti trisluoksnes išreikštines schemas pseudoparabolinei lygčiai su ivairiomis, tame tarpe nelokaliosiomis, kraštinėmis salygomis. Taip pat, pateikti skaitinių eksperimentų rezultatai.

Raktiniai žodžiai: pseudoparabolinė lygtis, trisluoksnė išreikštinė schema, tikrinių reikšmių uždavinys.