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An eigenvalue problem for the differential operator with an integral condition*

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Abstract. We analyze solution of a two-dimensional parabolic equation with a nonlocal integral condition by a locally one-dimensional method. The main aim of the paper is to deduce stability conditions of a system of one-dimensional equations with one integral condition. To this end, we analyze the structure of the spectrum of the differential operator with an integral condition.

Keywords: parabolic equation, nonlocal integral condition, eigenvalue problem, stability conditions.

Introduction

Let us analyze an eigenvalue problem of the system

$$\frac{d^2u_i}{dx^2} + \lambda u_i = 0, \quad i = 1, 2, \ldots, N-1, \tag{1}$$

with boundary conditions

$$u_i(0) = 0, \tag{2}$$

$$u_i(1) = \gamma_i h \sum_{k=1}^{N-1} \int_0^1 u_k(x)\, dx, \tag{3}$$

where $hN = 1$.

There are many papers devoted to the eigenvalue problem for ordinary differential operator with an integral or multi-point boundary condition ([1, 2, 3, 4, 5], see also bibliography in these articles). In this paper, we analyze an eigenvalue problem for system (1)–(3). This problem is realated with a two-dimensional parabolic equation

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with an integral boundary condition:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad x,y \in \Omega,\ 0 < t \leqslant T, \tag{4}$$

$$u(x,y,0) = \varphi(x,y), \quad x,y \in \Omega, \tag{5}$$

$$u(0,y,t) = \mu_1(y,t), \quad y \in \Omega,\ 0 < t \leqslant T, \tag{6}$$

$$u(1,y,t) = \mu_2(y,t), \quad y \in \Omega,\ 0 < t \leqslant T, \tag{7}$$

$$u(x,1,t) = \gamma_(x) \iint_\Omega u(x,y,t)\,dx\,dy + \mu_4(x,t), \quad x \in \Omega,\ 0 < t \leqslant T, \tag{8}$$

$$u(x,0,t) = \mu_3(x,t), \quad x \in \Omega,\ 0 < t \leqslant T, \tag{9}$$

where $\Omega = \{0 \leqslant x,y \leqslant 1\}$ is rectangular and $t \in [0,T]$.

Problem (4)–(9) solved by the finite difference method, i.e., it was reduced to a difference problem and then solved using one of the simplest methods, a locally one-domensional method. So

$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^n}{\tau} = \Lambda_1 u_{ij}^{n+\frac{1}{2}} + \frac{1}{2} f_{ij}^{n+1}, \tag{10}$$

$$u_{0j}^{n+\frac{1}{2}} = \mu_{1j}^{n+\frac{1}{2}}, \tag{11}$$

$$u_{Nj}^{n+\frac{1}{2}} = \mu_{2j}^{n+\frac{1}{2}}, \tag{12}$$

and

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\tau} = \Lambda_2 u_{ij}^{n+1} + \frac{1}{2} f_{ij}^{n+1}, \tag{13}$$

$$u_{i0}^{n+1} = \mu_{3i}^{n+1}, \tag{14}$$

$$u_{iN}^{n+1} = \gamma_i h^2 \sum_i \sum_j \rho_{ij} u_{ij}^{n+1} + \mu_{4i}^{n+1}, \tag{15}$$

where $\Lambda_1 u_{ij}^{n+\frac{1}{2}} = \frac{u_{i-1j}^{n+\frac{1}{2}} - 2u_{ij}^{n+\frac{1}{2}} + u_{i+1j}^{n+\frac{1}{2}}}{h^2}$, $\Lambda_2 u_{ij}^{n+1} = \frac{u_{i,j-1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j+1}^{n+1}}{h^2}$.

The stability of the second step (13)–(15) depends on the spectrum of the operator Λ_2

$$\frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h^2} + \lambda u_{ij} = 0, \quad i,j = 1,\dots,N-1, \tag{16}$$

$$u_{iN} = \gamma_i h^2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} u_{ij}, \tag{17}$$

$$u_{i0} = 0. \tag{18}$$

In this paper, we analyze problem (1)–(3). Problem (16)–(18) is a difference analogue of this differential problem.

1 The analysis of eigenvalues

As $\lambda = 0$, the solution of equation (16) is

$$u_i(x) = c_i x + c_2. \tag{19}$$

The solution with condition (2) is

$$u_i(x) = c_i x. \tag{20}$$

If we take the nonlocal boundary condition $u(1)$, we get

$$c_i = \gamma_i \frac{1}{N} \sum_{k=1}^{N-1} \int_0^1 c_i x \, dx. \tag{21}$$

Theorem 1. *The eigenvalue $\lambda = 0$ appears in problem* (1)–(3) *if and only if*

$$\sum_{i=1}^{N-1} \gamma_i = 2N. \tag{22}$$

Proof. Since

$$c_i = \gamma_i \frac{1}{N} \sum_{k=1}^{N-1} \int_0^1 c_i x \, dx, \tag{23}$$

by simplifying (23) we obtain

$$c_i = \gamma_i \frac{1}{2N} \sum_{k=1}^{N-1} c_i. \tag{24}$$

Thus, we obtain $N-1$ linear equations

$$\left(1 - \frac{\gamma_1}{2N}\right) c_1 - \frac{\gamma_1}{2N} c_2 - \cdots - \frac{\gamma_1}{2N} c_{N-1} = 0, \tag{25}$$

$$-\frac{\gamma_2}{2N} c_1 + \left(1 - \frac{\gamma_2}{2N}\right) c_2 - \cdots - \frac{\gamma_2}{2N} c_{N-1} = 0, \tag{26}$$

$$\vdots \tag{27}$$

$$-\frac{\gamma_{N-1}}{2N} c_1 - \frac{\gamma_{N-1}}{2N} c_2 + \cdots + \left(1 - \frac{\gamma_{N-1}}{2N}\right) c_{N-1} = 0. \tag{28}$$

There exists the nontrivial solution (21), if and only if the determinant of this system is equal to zero. Thus, we get that

$$1 - \sum_{i=1}^{N-1} \frac{\gamma_i}{2N} = 0. \tag{29}$$

As $\lambda < 0$, the general solution of equations (1), (2) is

$$u_i(x) = c_{i2} \sinh \beta x, \tag{30}$$

where $\beta = \sqrt{-\lambda}$, $\lambda = \beta^2$.

Since the general solution should satify nonlocal condition (3), after inserting it into (30) and doing arithmetical operations we, derive

$$c_i \sinh \beta = \gamma_i \frac{1}{N} \sum_{k=1}^{N-1} c_i \frac{\cosh(\beta) - 1}{\beta}. \tag{31}$$

So we obtain $N-1$ linear equations

$$\left(\sinh\beta - \frac{\gamma_1(\cosh\beta-1)}{\beta N}\right)c_1 - \frac{\gamma_1(\cosh\beta-1)}{\beta N}c_2 - \cdots - \frac{\gamma_1(\cosh\beta-1)}{\beta N}c_{N-1} = 0,$$
$$-\frac{\gamma_2(\cosh\beta-1)}{\beta N}c_1 + \left(\sinh\beta - \frac{\gamma_2(\cosh\beta-1)}{\beta N}\right)c_2 - \cdots - \frac{\gamma_2(\cosh\beta-1)}{\beta N}c_{N-1} = 0,$$
$$\vdots$$
$$-\frac{\gamma_{N-1}(\cosh\beta-1)}{\beta N}c_1 - \frac{\gamma_{N-1}(\cosh\beta-1)}{\beta N}c_2 + \cdots + \left(\sinh\beta - \frac{\gamma_{N-1}(\cosh\beta-1)}{\beta N}\right)c_{N-1} = 0.$$

Equating to zero the determinant of this system, we get the folowing equality

$$\sinh^{N-1}\beta - \sinh^{N-2}\beta\sum_{i=1}^{N-1}\frac{\gamma_i(\cosh\beta-1)}{\beta N} = 0. \tag{32}$$

Now we obtain two equations

$$\sinh^{N-2}\beta = 0, \tag{33}$$
$$\sinh\beta - \sum_{i=1}^{N-1}\frac{\gamma_i(\cosh\beta-1)}{\beta N} = 0. \tag{34}$$

From the first one we find the solution $\beta = 0$, but, under the assumption, we have that $\beta \neq 0$. That yields the second equation

$$\frac{\beta N}{\sum_{i=1}^{N-1}\gamma_i} = \tanh\frac{\beta}{2}. \tag{35}$$

Theorem 2. *Problem* (1)–(3) *has a negative eigenvalue, and it is only if*

$$\frac{1}{N}\sum_{i=1}^{N-1}\gamma_i > 2.$$

Proof. Let us take two functions

$$f_1(\beta) = \frac{\beta N}{\sum_{i=1}^{N-1}\gamma_i}, \tag{36}$$
$$f_2(\beta) = \tanh\frac{\beta}{2}, \tag{37}$$

They are equal with $\beta = 0$. Let us find the derivatives

$$f_1'(\beta) = \frac{N}{\sum_{i=1}^{N-1}\gamma_i} \quad \text{and} \quad f_2'(\beta) = \frac{1}{2}\left(1 - \tanh\frac{\beta}{2}\right), \tag{38}$$

By inserting $f_1'(0)$ and $f_2'(0)$ we attain

$$f_1'(0) = \frac{N}{\sum_{i=1}^{N-1} \gamma_i}, \qquad f_2'(0) = \frac{1}{2}. \tag{39}$$

Thus, if $\frac{1}{N}\sum_{i=1}^{N-1} \gamma_i < 2$, then the curves do not intersect, and if $\frac{1}{N}\sum_{i=1}^{N-1} \gamma_i > 2$, then the equation in the interval $(0, \infty)$ has one root.

Next, assume $\lambda > 0$ and denote that

$$\alpha = \sqrt{\lambda} > 0. \tag{40}$$

We get that the general solution of (1) with condition (2) is

$$u_i(x) = c_i \sin \alpha x. \tag{41}$$

We insert it into condition (3) and, after some arithmetical operations, we obtain that

$$c_i \sin \alpha = \gamma_i \frac{1 - \cos \alpha}{\alpha N} \sum_{k=1}^{N-1} c_k. \tag{42}$$

Consequently, we derive $N - 1$ linear equations once again:

$$\begin{aligned}
&\left(\sin \alpha - \frac{\gamma_1(1-\cos\alpha)}{N\alpha}\right)c_1 - \frac{\gamma_1(1-\cos\alpha)}{N\alpha}c_2 - \cdots - \frac{\gamma_1(1-\cos\alpha)}{N\alpha}c_{N-1} = 0,\\
&\frac{\gamma_2(1-\cos\alpha)}{N\alpha}c_2 + \left(\sin \alpha - \frac{\gamma_2(1-\cos\alpha)}{N\alpha}\right)c_2 - \cdots - \frac{\gamma_2(1-\cos\alpha)}{N\alpha}c_{N-1} = 0,\\
&\vdots\\
&-\frac{\gamma_{N-1}(1-\cos\alpha)}{N\alpha}c_1 - \frac{\gamma_{N-1}(1-\cos\alpha)}{N\alpha}c_2 + \cdots\\
&\quad + \left(\sin \alpha - \frac{\gamma_{N-1}(1-\cos\alpha)}{N\alpha}\right)c_{N-1} = 0.
\end{aligned}$$

Now we are interested when the determinant of this system of linear equations is equal to zero. Hence we derive that

$$\sin^{N-1}\alpha - \sin^{N-2}\alpha \sum_{i=1}^{N-1} \frac{\gamma_i(1-\cos\alpha)}{N\alpha} = 0. \tag{43}$$

Thus we get two equations

$$\sin^{N-2}\alpha = 0, \tag{44}$$

$$\sum_{i=1}^{N-1} \gamma_i = \frac{N\alpha \sin\alpha}{1-\cos\alpha}. \tag{45}$$

Theorem 3. *With all the values of γ_i equation (1)–(3) has $N - 2$ multiple roots ($\alpha > 0$) and they are independent of γ_i and infinitely many positive eigenvalues depend on γ_i.*

Proof. The roots of equation (43)

$$\alpha_k = \pi k, \quad k = 1, 2, \dots \tag{46}$$

are independent of γ_i. We need to find the roots of equation (44). Let us choose

$$f_1(\alpha) = \frac{\sum_{i=1}^{N-1} \gamma_i}{\alpha N}, \tag{47}$$

$$f_2(\alpha) = \frac{\sin \alpha N \alpha}{1 - \cos \alpha} = \tan \frac{\alpha}{2}. \tag{48}$$

It is obvious that with each value γ_i of equation (46), we get very many eigenvalues (which depends on the length of the interval we explored).

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REZIUMĖ

Tikrinių reikšmių uždavinys diferencialiniam operatoriui su integraline sąlyga
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Mes analizuojame sprendimą dvimatės parabolinės lygties su nelokaliąja integraline sąlyga lokaliai-vienmačiu metodu. Darbo pagrindinis tikslas yra išvesti stabilumo sąlygas sistemai vienamačių lygčių su integalinėmis sąlygomis. Siekiant šio tikslo, mes analizuojame spektro struktūrą skirtuminio operatoriaus su nelokaliąja integraline sąlyga.

Raktiniai žodžiai: parabolinė lygtis, nelokalioji integralinė sąlyga, Tikrinių reikšmių uždavinys, stabilumo sąlygos.