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Clocking convergence to Arnold tongues – the circle map revisited*

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Abstract. Computational techniques based on ranks of Hankel matrices is used to study the convergence to Arnold tongues. It appears that the process of convergence to the phase-locked mode is far from being trivial. The stable, the unstable and the manifold of non-asymptotic convergence intertwine in the parameter plane of the circle map. Pseudoranks of Hankel matrices carry important physical information about transient processes taking place in discrete nonlinear iterative maps. These pictures in the parameter plane are also beautiful from the aesthetical point of view.

Keywords: the circle map, Arnold tongue, rank of a sequence.

Introduction

Clocking convergence is an important tool for investigating various aspects of iterative (chaotic) maps. The rate of convergence to the critical attractor when an ensemble of initial conditions is uniformly spread over the entire phase space may provide the insight into the fractal nature and the scale invariance of the dynamical attractor [3, 7]. Numerical convergence of the discrete logistic map gauged with a finite computational accuracy is investigated in [2] where forward iterations are used to identify self-similar patterns in the region before the onset to chaos. An alternative technique based on the concept of the Hankel rank is proposed in [5] for clocking the convergence of iterative chaotic maps.

The insight into the embedded algebraic complexity of the nonlinear system is revealed by computing and visualizing of Hankel ranks in the space of system's parameters and initial conditions. It is shown in [5] that the computation of Hankel ranks can be effectively used to identify and assess the sensitivity of nonlinear systems to initial conditions and can be used as a simple and effective numerical tool for qualitative investigation of the onset of chaos for discrete nonlinear iterative maps.

We will use the discrete iterative circle map to illustrate the process of convergence to stationary states. The circle map is a paradigmatic model of a nonlinear iterative dynamical system used to study the dynamical behavior of a beating heart [4]. We will show that the study of the convergence rate to a periodic orbit of a single circle map can produce beautiful and appealing patterns. Moreover, these graphical

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pictures contain important information on the stability of periodic orbits of the circle map. This information could be useful whenever the manipulation or control of quasiperiodic nonlinear systems would be considered.

The paper is organized as follows. The algorithm for the computation of the Hankel rank of a sequence and its properties are investigated in Section 1. The circle map, the convergence to Arnold tongues and patterns of pseudoranks are explored in Section 2. The numerical computation of pseudoranks is discussed in Section 3 and concluding remarks are given in the last section.

1 The concept of the Hankel rank of a sequence

The algorithm for the computation of the Hankel rank of a sequence $\{p_j\}$, $j = 0, 1, \dots,$ $p_j \in \mathbb{R}$, has been introduced in [5]. A sequence of Hankel matrices reads:

$$H_n := (p_{i+j-2})_{1\leqslant i,j\leqslant n} = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ \cdots & \cdots & \cdots & \cdots \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{bmatrix},$$

$n = 1, 2, \dots.$ The Hankel transform (the sequence of determinants of Hankel matrices) $\{d_n\}$, $n = 0, 1, \dots,$ reads:

$$d_n := \det H_n, \quad n = 1, 2, \dots.$$

The sequence $\{p_j\}$, $j = 0, 1, \dots,$ has an H-rank $Hr\{p_j\} = m$, $j = 0, 1, \dots,$ $m \in \mathbb{Z}_0$, $m < +\infty$; if the sequence of determinants of Hankel matrices has the following structure: $\{d_1, d_2, \dots, d_m, 0, 0, \dots\}$ where $d_m \neq 0$ and $d_{m+1} = d_{m+2} = \cdots = 0$.

It is admited that $Hr\{0, 0, \dots\} = 0$. Note that $Hr\{p_0, \dots, p_m, 0, 0, \dots\} = m + 1$, if only $p_m \neq 0$ for $m = 0, 1, 2, \dots.$

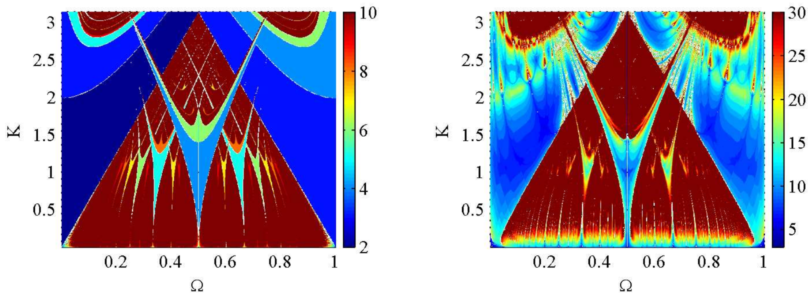
2 The circle map

The circle map is represented by the one-dimensional iterative map:

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi} \cdot \sin(2\pi\theta_n), \tag{1}$$

where θ_n value lies between 0 and 1 ($2\pi\theta_n$ is a polar angle); K is the coupling strength; Ω is the driving phase and $n = 0, 1, \dots.$ The circle map exhibits a phenomenon called phase locking if small to intermediate values of K ($0 < K < 1$) and certain values of Ω are considered. In a phase-locked region, the values θ_n advance as a rational multiple of n. The phase-locked regions in $\Omega - K$ parameter plane are called Arnold tongues [1].

The main objective of this paper is to present the rank of a Hankel matrix as a tool for clocking the convergence to phase-locked regions of the circle map. The measurements of the convergence rate to Arlond tongues can reveal important physical information on the properties of the iterative system. Moreover, such enriched representation of Arnold tongues produces aesthetically beautiful pictures.



(A) $k = 4000$, $\overline{m} = 10$.

(B) $k = 0$, $\overline{m} = 30$.

Fig. 1. The pseudoranks of the circle map in the space of system's parameters.

2.1 The convergence to Arnold tongues

We use the Hankel rank as the computational tool for the reconstruction of Arnold tongues. We compute Hankel ranks in the region $0 \leqslant \Omega \leqslant 1$ and $0 \leqslant K \leqslant \pi$. For every pair of Ω and K we start the iterative process, omit $k = 4000$ forward iterations (in order to cease down the transients), construct the sequence $\{\theta_{j+k}\}$, $j = 0, 1, \ldots$, and calculate the Hankel rank of that sequence. As shown in [5], the Hankel rank of a chaotic sequence does not exist (the Hankel rank tends to infinity then). Therefore we set the upper limit for the Hankel rank $\overline{m} = 10$. If the sequence of determinants does not vanish until $m = 10$ we terminate the process assuming that $Hr\{p_{j+k}\} = \overline{m}$; $j = 0, 1, \ldots$. The results are shown in Fig. 1(A). Zones where the phase locking occurs are clearly separated by red insets. The well-known shape of Arnold tongues in the circle map [6] is clearly seen in resulting picture.

The computational experiment is repeated without omitting transient processes ($k = 0$). The initial condition θ_0 is set to 0.5; the results are shown in Fig. 1(B). It could be proven that the Hankel rank of the process with transient processes is higher compared to the the Hankel rank of the same process but with transient processess omitted. Such an effect can be clearly seen in Fig. 1(B). The research showed that the higher value of the upper bound ($\overline{m} = 30$) helps to visualize complex transient processes in regions where phase locking occurs (Fig. 1(B)). The interesting pattern in the blue colored zones is rather unexpected and requires additional attention.

3 The computation of pseudoranks

Theoretically one needs to find such matrix dimension that the determinant of the Hankel matrix is equal to zero. But in practice it is sufficient to compute determinants up to a certain precision, like the machine epsilon. Thus we continue the computation of determinants until $|\tilde{H}_m| < \varepsilon$. In this respect our computations reveal not the rank, but the estimate of the rank of a sequence.

The selection of a particular value of ε requires additional attention. As mentioned previously, the structure of Arnold tongues in the circle map is well-known. Thus we select Fig. 1(B), fix the parameter K ($K = 0.4\pi$) and cut through the map of pseudoranks by varying the parameter Ω. Moreover, we perform the computation of the estimates of the ranks for different initial conditions ($0 \leqslant \theta_0 \leqslant 1$) for different ε. Results are illustrated in Fig. 2. The evolution of interesting patterns of pseudoranks

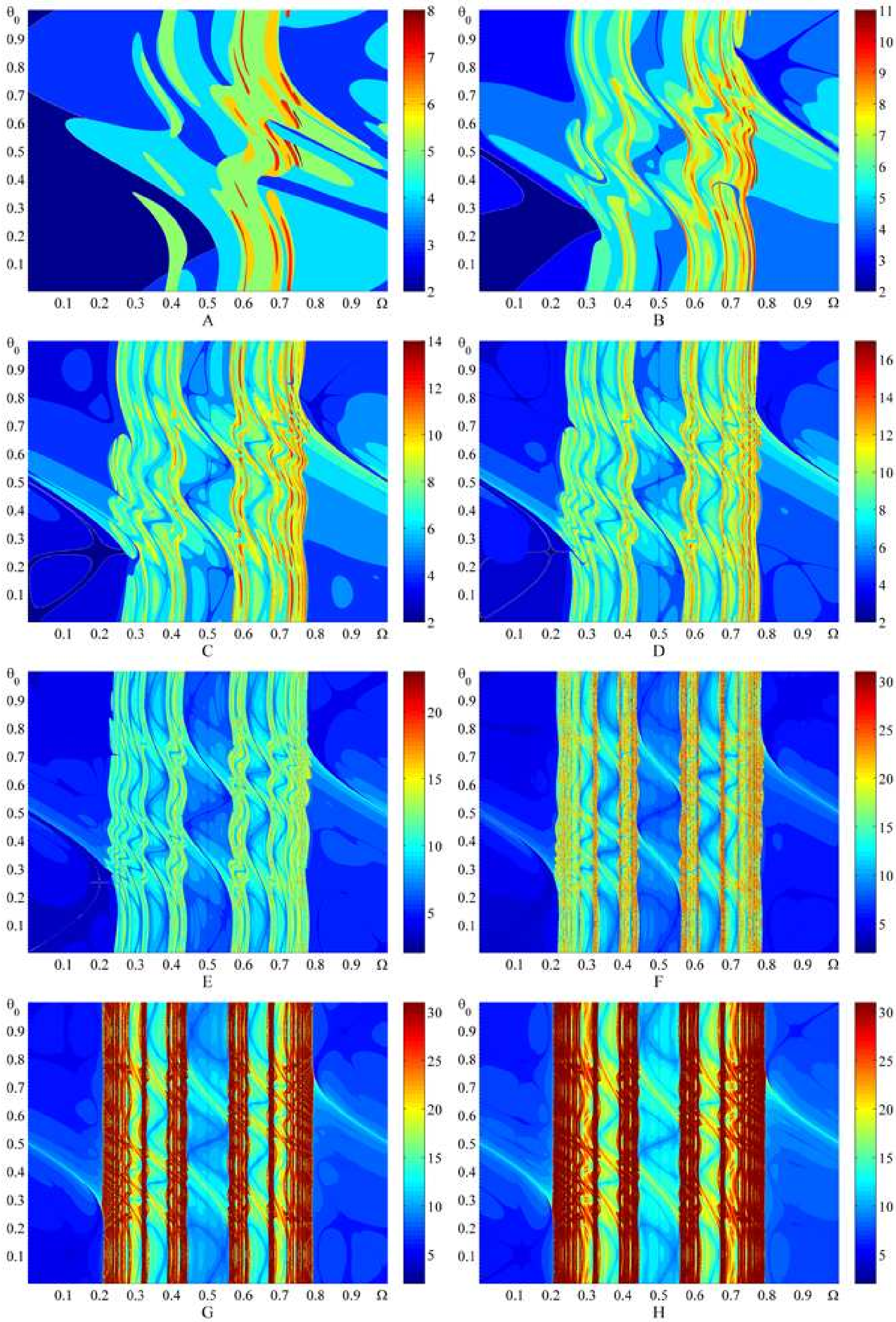


Fig. 2. Maps of pseudoranks for different initial conditions ($0 \leqslant \theta_0 \leqslant 1$) at A: $\varepsilon = 10^{-1}$; B: $\varepsilon = 10^{-2}$; C: $\varepsilon = 10^{-3}$; D: $\varepsilon = 10^{-4}$; E: $\varepsilon = 10^{-5}$; F: $\varepsilon = 10^{-10}$; G: $\varepsilon = 10^{-20}$; H: $\varepsilon = 10^{-30}$.

can be observed as the value of ε is decreased. Note that the maximum rank in colorbars is detected automatically and depends on ε.

A naked eye cannot see principal differences between Fig. 2(G) and Fig. 2(H). Thus we fix $\varepsilon = 10^{-30}$ and use this value for the research (all previous figures of pseudoranks are constructed using this value of ε).

4 Conclusions

1. The complexity of transient processes poses an important feature for the applicability of control techniques applicable for the circle map. Let us consider a situation when the circle map is operating in a chaotic regime and one needs to bring it to the phase-locked regime. One may determine the phase of the circle map immediately before the jump and then perform the jump from the chaotic region into the phase-locked region in the parameter plane of the circle map in one forward iteration by controlling one or both parameters of the circle map. Such control method would ensure that the system will start operating in the phase-locked mode immediately after the correction of its parameters (provided the distribution of manifolds on Arnold tongues is known).

2. The value of $\varepsilon = 10^{-30}$ is selected to be small enough to produce visually informative patterns in the plane of system's parameters and initial conditions. Thus we start at relatively high value ($\varepsilon = 10^{-1}$) and continually reduce it till there is no visual difference in the plane of system's parameters.

3. The development of such control strategies is a definite object of future research. But pictures representing the manifold of non-asymptotic convergence do carry not only an important physical information but are also aesthetically beautiful.

4. Further research showed that the computation and visualization of Hankel ranks reveals three manifolds of the discrete iterative map: the stable manifold, the unstable manifold and the manifold of the non-asymptotic convergence.

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REZIUMĖ

Konvergavimo prie Arnoldo liežuvių apskritiniame vaizdavime greičio įvertinimas

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Konvergavimo greitis prie Arnoldo liežuvių tiriamas skaitiniais metodais, paremtais Hankelio matricos rangu. Konvergavimo pobūdis į sinchronizuotą režimą nėra trivialus. Apskritiminio žemėlapio parametrų plokštumoje susipina stabili, nestabili ir neasimptotinio konvergavimo daugdaros. Hankelio matricų rangų įverčiai parodo svarbią fizikinę informaciją apie pereinamuosius procesus netiesiniuose iteraciniuose procesuose. Parametrų plokštumos vaizdai taip pat yra estetiškai gražūs.

Raktiniai žodžiai: apskritiminis vaizdavimas, Arnoldo liežuvis, sekos rangas.