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A weighted limit theorem for periodic Hurwitz zeta-function

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Abstract. In the paper, a weighted limit theorem for weakly convergent probability measures on the complex plane for the periodic Hurwitz zeta function is obtained
Keywords: periodic Hurwitz zeta function, probability measure, weak convergence.

Introduction

Let $\mathfrak{a} = \{a_m\colon m \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a periodic with the least period $k \in \mathbb{N}$ sequence of complex numbers, and $\alpha \in (0,1]$ be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by Dirichlet series

$$\zeta(s,\alpha,\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

For $\sigma > 1$, the periodicity of $\mathfrak{a}$ implies the equality

$$\zeta(s,\alpha;\mathfrak{a}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{\alpha+l}{k}\right), \tag{1}$$

where

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \quad \sigma > 1,$$

is the classical Hurwitz zeta-function. Since the function $\zeta(s,\alpha)$ has a simple pole at $s = 1$ with residue 1, equality (1) gives analytic continuation for $\zeta(s,\alpha;\mathfrak{a})$ to the whole complex plane, except maybe, for a simple pole at $s = 1$. If

$$a := \frac{1}{k} \sum_{l=0}^{k-1} a_l = 0,$$

then the function $\zeta(s,\alpha;\mathfrak{a})$ is entire, while, in the case $a \neq 0$, the point $s = 1$ is a simple pole with residue a.

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and by $\operatorname{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Suppose that α is transcendental. Then in [2], by the way, it was obtained that, for $\sigma > \frac{1}{2}$, probability measure

$$\frac{1}{T}\operatorname{meas}\bigl\{t \in [0,T]\colon \zeta(\sigma+it,\alpha;\mathfrak{a}) \in A\bigr\},\ A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$.

The aim of this note is to prove a weighted limit theorem on the complex plane for the function $\zeta(s,\alpha;\mathfrak{a})$. Let $w(t)$ be a positive function of bounded variation on $[T_0,\infty]$, $T_0>0$, such that

$$\lim_{T\to\infty} U(T,w) = \lim_{T\to\infty} \int_{T_0}^{T} w(t)\,dt = +\infty.$$

Also, we require that, for $\sigma>\frac{1}{2}$, $\sigma\neq 1$, and all $v\in\mathbb{R}$, the estimate

$$\int_{T_0+v}^{T+v} w(u-v)\bigl|\zeta(\sigma+it,\alpha,\mathfrak{a})\bigr|^2\,dt \ll U\bigl(1+|v|\bigr) \tag{2}$$

should be satisfied. Denote by $\mathbb{I}_A(t)$ the indicator function of a set A, and define the probability measure

$$P_{T,\sigma,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t)\mathbb{I}_{\{t:\zeta(\sigma+it,\alpha,\mathfrak{a})\in A\}}\,dt, \quad A\in\mathcal{B}(\mathbb{C}).$$

Theorem 1. *Suppose that α is transcendental, $\sigma>\frac{1}{2}$, and that the weight function satisfies the condition* (2)*. Then, on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$, there exists a probability measure P_σ such that the measure $P_{T,\sigma,w}$ converges weakly to P_σ as $T\to\infty$.*

1 Auxiliary rezults

We start with a weighted limit theorem on the infinite-dimensional torus. Let

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m=\{s\in\mathbb{C}\colon |s|=1\}$ for all $m\in\mathbb{N}_0$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega,\mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. Define the probability measure

$$Q_{T,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t)\mathbb{I}_{\{t:((m+\alpha)^{-it}:m\in\mathbb{N}_0)\in A\}}\,dt, \quad A\in\mathcal{B}(\Omega).$$

Lemma 1. *Suppose that α is transcendental. Then the probability measure $Q_{T,w}$ converges weakly to m_H as $T\to\infty$.*

Proof. Denote by $\mathbb{Z}$ the set of all integers. Then the dual group (the character group) of Ω is isomorphic to $\mathbb{D}=\bigoplus_{m=0}^{\infty}\mathbb{Z}_m$, where $\mathbb{Z}_m=\mathbb{Z}$ for all $m\in\mathbb{N}_0$. Let $\omega(m)$ be the projection of $\omega\in\Omega$ to the coordinate space γ_m, $m\in\mathbb{N}_0$.

An element $\underline{k}=(k_0,k_1,\dots)\in\mathbb{D}$, where only a finite number of integers k_m, $m\in\mathbb{N}_0$, are distinct from zero, acts on Ω by

$$\omega\to\omega^{\underline{k}} = \prod_{m=0}^{\infty}\omega^{k_m}(m), \quad \omega\in\Omega.$$

Therefore, the Fourier transform $g_{T,w}(\underline{k})$ of the measure $Q_{T,w}$ is

$$g_{T,w}(\underline{k}) = \int_{\Omega} \left(\prod_{m=0}^{\infty} \omega^{k_m}(m) \right) dQ_{T,w} = \frac{1}{U} \int_{T_0}^{T} w(t) \prod_{m=0}^{\infty} (m+\alpha)^{-itk_m}\, dt. \qquad (3)$$

Since α is transcendental, the set $\{\log(m+\alpha)\colon m \in \mathbb{N}_0\}$ is linearly independent over the field of rational numbers. Therefore, $\sum_{m=0}^{\infty} k_m \log(m+\alpha) = 0$ if and only if $\underline{k} = \underline{0}$. If $\underline{k} \neq \underline{0}$, then we have

$$\begin{aligned}
&\int_{T_0}^{T} w(t) \prod_{m=0}^{\infty} (m+\alpha)^{-itk_m}\, dt \\
&\quad = \int_{T_0}^{T} w(t)\, de^{-it\sum_{m=0}^{\infty} k_m \log(m+\alpha)}\, dt \\
&\quad = \left(-i \sum_{m=0}^{\infty} k_m \log(m+\alpha) \right)^{-1} \int_{T_0}^{T} w(t)\, de^{-it\sum_{m=0}^{\infty} k_m \log(m+\alpha)} \\
&\quad = O\left(\left| \sum_{m=0}^{\infty} k_m \ln(m+\alpha) \right|^{-1} \right),
\end{aligned}$$

where, as above, only a finite number of integers k_m are distinct from zero. This, together with (3), shows that

$$\lim_{T\to\infty} g_{T,w}(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}, \end{cases}$$

and the lemma follows from a continuity theorem on compact groups.

Now let $\sigma_1 > \frac{1}{2}$ be fixed, and, for $m, n \in \mathbb{N}_0$,

$$v_n(m,\alpha) = e^{-(\frac{m+\alpha}{n+\alpha})^{\sigma_1}}.$$

Then it is easy to show that the series

$$\zeta(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m v_n(m,\alpha)}{(m+\alpha)^s} \qquad (4)$$

converges absolutely for $\sigma > \frac{1}{2}$. Consider the probability measure

$$P_{T,n,\sigma,w}(A) = \frac{1}{u} \int_{T_0}^{T} w(t) \mathbb{I}_{\{t:\zeta_n(\sigma+i\cdot t,\alpha,\mathfrak{a})\in A\}}\, dt, \quad A \in \mathcal{B}(\mathbb{C}).$$

Lemma 2. *Suppose that α is transcendental and $\sigma > \frac{1}{2}$. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_{n,\sigma}$ such that the measure $P_{T,n,\sigma,w}$ converges weakly to $P_{n,\sigma}$ as $T \to \infty$.*

Proof. Define the function $h_{n,\sigma} : \Omega \to \mathbb{C}$ by the formula

$$h_{n,\sigma}(\omega) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m,\alpha)}{(m+\alpha)^{\sigma}}, \quad \omega \in \Omega.$$

The absolute convergence of the series (4) implies the continuity of the function $h_{n,\sigma}$. Since

$$h_{n,\sigma}\big(\big((m+\alpha)^{-it}\colon m\in\mathbb{N}_0\big)\big)=\zeta_n(\sigma+it,\alpha;\mathfrak{a}),$$

hence, using Theorem 5.1 from [1] and Lemma 2 we obtain that the measure $P_{T,n,\sigma,w}$ converges weakly to $m_H h_{n,\sigma}^{-1}$ as $T\to\infty$.

For the proof of Theorem 1, it remains to pass from the function $\zeta_n(s,\alpha;\mathfrak{a})$ to $\zeta(s,\alpha;\mathfrak{a})$. For this, the following statement will be applied.

Lemma 3. *Suppose that* $\sigma>\frac{1}{2}$, *and the condition* (1) *holds. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U}\int_{T_0}^{T}w(t)\big|\zeta(\sigma+it,\alpha;\mathfrak{a})-\zeta_n(\sigma+it,\alpha;\mathfrak{a})\big|\,dt=0.$$

Proof. The function $\zeta_n(s,\alpha;\mathfrak{a})$ can be written in the form

$$\zeta_n(s,\alpha;\mathfrak{a})=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\zeta(s+z,\alpha;\mathfrak{a})l_n(z,\alpha)\frac{dz}{z},$$

where

$$l_n(z,\alpha)=\frac{z}{\sigma_1}\Gamma\left(\frac{z}{\sigma_1}\right)(n+\alpha)^z,$$

and $\Gamma(s)$ denotes the Euler gamma function. From this, using the residue theorem, we derive that

$$\zeta_n(s,\alpha;\mathfrak{a})=\frac{1}{2\pi i}\int_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty}\zeta(s,\alpha;\mathfrak{a})l_n(z,\alpha)\frac{dz}{z}+\zeta(s,\alpha;\mathfrak{a})+a\frac{l_n(1-s,\alpha)}{1-s},$$

where $\sigma_2>\sigma_1$, and $\sigma_2<\sigma$. Therefore, as $T\to\infty$,

$$\begin{aligned}&\frac{1}{U}\int_{T_0}^{T}w(t)\big|\zeta(\sigma+it,\alpha;\mathfrak{a})-\zeta_n(\sigma+it,\alpha;\mathfrak{a})\big|\,dt\\&\quad\ll\int_{-\infty}^{\infty}\big|l_n(\sigma_2-\sigma+iv,\alpha)\big|\left(\int_{T_0+v}^{T+v}w(t-v)\big|\zeta(\sigma,\alpha,\mathfrak{a})\big|\,dt\right)dv+O\big(e^{-c|T|}\big).\end{aligned}\tag{5}$$

In view of (2), we find that

$$\begin{aligned}&\frac{1}{U}\int_{T_0+v}^{T+v}w(t-v)\big|\zeta(\sigma+it,\alpha;\mathfrak{a})\big|\,dt\\&\quad\ll\frac{1}{U}\left(\int_{T_0+v}^{T+v}w(t-v)\,dt\right)^{\frac{1}{2}}\left(\int_{T_0+v}^{T+v}w(t-v)\big|\zeta(\sigma+i\cdot t,\alpha,\mathfrak{a})\big|^2\,dt\right)^{\frac{1}{2}}\\&\quad\ll\big(1+|v|\big).\end{aligned}$$

Thus, by (5),

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{U}\int_{T_0}^{T}w(t)\big|\zeta(\sigma+it,\alpha;\mathfrak{a})-\zeta_n(\sigma+it,\alpha;\mathfrak{a})\big|\,dt\\&\quad\ll\int_{-\infty}^{\infty}\big|l_n(\sigma_2-\sigma+iv;\alpha)\big|\big(1+|v|\big)\,dv.\end{aligned}\tag{6}$$

Since $\sigma_2 - \sigma < 0$, the definition of $l_n(z, \alpha)$ shows that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \bigl|l_n(\sigma_2 - \sigma + iv, \alpha)\bigr| \bigl(1 + |v|\bigr)\, dv = 0,$$

and the lemma follows from (6).

2 Proof Theorem 1

Now we a ready to prove Theorem 1. First we observe that family of probability measures $\{P_{n,\sigma}\colon n \in \mathbb{N}\}$, where $P_{n,\sigma}$ is the limit measure in Lemma 2, is tight. Really, for arbitrary $M > 0$,

$$\frac{1}{U}\int_{T_0}^{T} w(t)\mathbb{I}_{\{t:|\zeta_n(\sigma+it,\alpha;\mathfrak{a})|>M\}}\, dt \ll \frac{1}{MU}\int_{T_0}^{T} w(t)\bigl|\zeta_n(\sigma + it, \alpha; \mathfrak{a})\bigr|\, dt. \tag{7}$$

Moreover, by (2) and Lemma 3,

$$\begin{aligned} &\sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t)\bigl|\zeta_n(\sigma + it, \alpha; \mathfrak{a})\bigr|\, dt \\ &\quad \ll 1 + \limsup_{T\to\infty} \frac{1}{U}\biggl(\int_{T_0}^{T} w(t)\, dt \int_{T_0}^{T} w(t)\bigl|\zeta(\sigma + it, \alpha; \mathfrak{a})\bigr|\, dt\biggr)^{\frac{1}{2}} \ll R < \infty. \end{aligned} \tag{8}$$

Now let $M = M_\varepsilon = R\varepsilon^{-1}$, where $\varepsilon > 0$ is arbitrary number. Then (7), (8) and Theorem 2.1 of [1] give, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} P_{n,\sigma}\bigl(\{s \in \mathbb{C}\colon |s| > M_\varepsilon\}\bigr) &\leqslant \liminf_{T\to\infty} P_{T,n,\sigma,w}\bigl(\{s \in \mathbb{C}\colon |s| > M_\varepsilon\}\bigr) \\ &= \liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t)\mathbb{I}_{\{t\colon\ |\zeta_n(\sigma+it,\alpha;\mathfrak{a})|>M_\varepsilon\}}\, dt \\ &\leqslant \limsup_{T\to\infty} \frac{1}{T}\int_{T_0}^{T} w(t)\mathbb{I}_{\{t\colon\ |\zeta_n(\sigma+it,\alpha;\mathfrak{a})|>M_\varepsilon\}}\, dt \leqslant \varepsilon. \end{aligned} \tag{9}$$

Define $K_\varepsilon = \{s \in \mathbb{C}\colon |s| \leqslant M_\varepsilon\}$. Then the set K_ε is compact, and, in view of (9), for all $n \in \mathbb{N}_0$,

$$P_{n,\sigma}(K_\varepsilon) \geqslant 1 - \varepsilon.$$

The tightness of $\{P_{n,\sigma}\colon n \in \mathbb{N}\}$ implies its relative compactness. Therefore, there exists a sequence $\{P_{n_k,\sigma}\} \subset \{P_{n,\sigma}\}$ such that $P_{n_k,\sigma}$ converges weakly to a certain probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $k \to \infty$. Let $\theta = \theta_T$ be a random variable on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$ having the distribution

$$\mathbb{P}(\theta \in A) = \frac{1}{U}\int_{T_0}^{T} w(t)\mathbb{I}_A\, dt, \quad A \in \mathcal{B}(\mathbb{C}).$$

Suppose that $X_n(\sigma)$ is a complex-valued random variable with the distribution $P_{n,\sigma}$. Then the above remark implies the relation

$$X_{n_k}(\sigma) \xrightarrow[k\to\infty]{\mathcal{D}} P_\sigma, \tag{10}$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution. Define

$$X_{T,n}(\sigma) = \zeta_n(\sigma + i\theta_T, \alpha; \mathfrak{a}).$$

Then, by Lemma 2,

$$X_{T,n}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} X_n(\sigma). \tag{11}$$

Putting $X_T(\sigma) = \zeta(\sigma + i\theta_T, \alpha; \mathfrak{a})$, we have from Lemma 3 that, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\big|X_T(\sigma) - X_{T,n}(\sigma)\big| \geqslant \varepsilon\big) \\ &\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U\varepsilon} \int_{T_0}^{T} w(t)\big|\zeta(\sigma + it, \alpha; \mathfrak{a}) - \zeta_n(\sigma + it, \alpha; \mathfrak{a})\big|\, dt = 0. \end{aligned}$$

This, (10), (11), and Theorem 4.2 of [1] now show that

$$X_T(\sigma) \xrightarrow[T\to\infty]{D} P_\sigma.$$

References

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REZIUMĖ

Ribine teorema periodinems Hurvico dzeta funkcijoms su svoriu

Oleg Lukašonok

Straipsnyje yra pateikiama ribinė teorema su svoriu periodinei Hurvico dzeta funkcijai. Teoremos įrodyme yra panaudojamas tikimybinių matų silpnasis konvergavimas kompleksinėje plokštumoje.

Raktiniai žodžiai: periodinė Hurvico dzeta funkcija, tikimybinis matas, silpnasis konvergavimas.