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A transformation formula with primitive character

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Abstract. We prove a transformation formula for the exponential sum involving the divisor function, and primitive character modulo q. This formula can be applied to obtain meromorphic continuation for the Mellin transform of the square of Dirichlet L-function with primitive character.

Keywords: Dirichlet L-function, Estermann zeta-function, Mellin transform.

1 Introduction

Let $s = \sigma + it$ be a complex variable, χ be a Dirichlet character modulo q, $q > 1$, and let

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

denote the corresponding Dirichlet L-function. For the investigation of the mean square of L-functions, the modified Mellin transform

$$\mathcal{Z}_1(s, \chi) = \int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 x^{-s} \, dx, \quad \sigma > 1,$$

can be applied.

For a meromorphic continuation of the transform $\mathcal{Z}_1(s, \chi)$, a certain transformation formula is needed. Similar transformation formulae are also used in the case of the Riemann zeta-function see [3, 4]. A transformation formula for $\mathcal{Z}_1(s, \chi)$ with principal character has been obtained in [1]. The aim of this note is to obtain a transformation formula with primitive character.

Let $G(\chi)$ denote the Gauss sum, i.e.,

$$G(\chi) = \sum_{a=1}^{q} \chi(a) e^{2\pi i a/q}.$$

As usual, denote by $d(m)$ the divisor function

$$d(m) = \sum_{d|m} 1,$$

and let γ be the Euler constant. For $\operatorname{Re} z > 0$, $\operatorname{Im} z \neq 0$, define

$$\Phi(z;\chi,q) = \sum_{m=1}^{\infty} d(m)\chi(m)e^{2\pi i m/q}e^{-mz} - \frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\frac{(q,a+1)}{qz}\left(\gamma - 2\log\frac{q}{(q,a+1)} - \log z\right).$$

In this paper, we prove a formula for $\Phi(z^{-1};\chi,q)$.

Let a_{0a}^{+} and a_{0a}^{-} be the constant terms in the Laurent series expansion at $s=1$ for the Estermann zeta-functions $E\bigl(s;\overline{(\frac{a+1}{(q,a+1)})}/\frac{q}{(q,a+1)},0\bigr)$ and $E\bigl(s;-\overline{(\frac{a+1}{(q,a+1)})}/\frac{q}{(q,a+1)},0\bigr)$, respectively, where $\overline{(\frac{a+1}{(q,a+1)})}/\frac{q}{(q,a+1)}$ is connected to $\frac{a+1}{(q,a+1)}/\frac{q}{(q,a+1)}$ by the congruence $\frac{a+1}{(q,a+1)}\overline{(\frac{a+1}{(q,a+1)})} \equiv 1\bigl(\operatorname{mod}\ \frac{q}{(q,a+1)}\bigr)$, $a=1,\dots,q$. Moreover, denote

$$\delta = \begin{cases} 1 & \text{if } \operatorname{Im} z > 0,\\ -1 & \text{if } \operatorname{Im} z < 0. \end{cases}$$

and, for $1 < b < 2$, define

$$I(z,b) = \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{q}\right)^{1-2w}\Gamma(w)\Biggl\{\sin^{-1}(\pi w)E\left(w;\frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}},0\right) \times (q,a+1)^{1-2w} + \cot(\pi w)\left(w;-\frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}},0\right)(q,a+1)^{1-2w} + \delta i E\left(w;-\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\right)\Biggr\}z^{1-w}\,dw.$$

Theorem 1. *Suppose that* $\operatorname{Re} z > 0$ *and* $\operatorname{Im} z \neq 0$. *Then*

$$\Phi(z^{-1};\chi,q) = -\frac{2\pi i\delta z}{q}\sum_{m=1}^{\infty} d(m)\chi(m)e^{-2\pi i m/q}e^{-\frac{4\pi^2 m z}{q^2}} + \frac{q}{2\pi^2 G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a+1)}(a_{0a}^{+} - a_{0a}^{-}) + I(z,b).$$

2 Proof of the transformation formula

Firstly, similarly as in [4], we express $\sum_{m=1}^{\infty} d(m)\chi(m)\exp\{2\pi i m/q\}m^{-s}$ by the Gaussian sum and Estermann zeta-function:

$$\sum_{m=1}^{\infty} d(m)\chi(m)\exp\{2\pi i m/q\}m^{-s} = \frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)E\left(s;\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\right).$$

Then, using the Mellin transformation formula, we find

$$\begin{aligned}\sum_{m=1}^{\infty} & d(m)\chi(m)\exp\{2\pi i m/q\}e^{-mz}\\ &= \sum_{m=1}^{\infty} d(m)\chi(m)\exp\{2\pi i m/q\}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\Gamma(w)(mz)^{-w}\,dw\\ &= \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\Gamma(w)z^{-w}\sum_{m=1}^{\infty} d(m)\chi(m)\exp\{2\pi i m/q\}m^{-w}\,dw\\ &= \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int\limits_{2-i\infty}^{2+i\infty}\Gamma(w)E\Bigg(w;\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Bigg)z^{-w}\,dw. \qquad (1)\end{aligned}$$

From the Laurent series expansion for Estermann zeta-function at the point $w = 1$ (see in [2]), we have

$$E\Bigg(w;\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Bigg) = \frac{(q,a+1)}{q}\Bigg(\frac{1}{(w-1)^2}+\frac{2\gamma-2\log\frac{q}{(q,a+1)}}{w-1}\Bigg)+a_{0a}+\cdots. \qquad (2)$$

From this, using the expansions

$$\Gamma(w) = 1-\gamma(w-1)+\frac{\Gamma''(1)(w-1)^2}{2}+\cdots, \qquad (3)$$

$$z^{-w} = z^{-1}e^{-(w-1)\log z} = z^{-1}\Bigg(1-\Big(w-1\Big)\log z+\frac{(w-1)^2\log^2 z}{2}+\cdots\Bigg), \qquad (4)$$

we find

$$\operatorname*{Res}_{w=1}\Gamma(w)E\Bigg(w;\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Bigg)z^{-w} = \frac{(q,a+1)}{qz}\Bigg(\gamma-2\log\frac{q}{(q,a+1)}-\log z\Bigg).$$

Now, moving the line of integration in (1) to the left and having in mind that $0 < c < 1$, from the definition of $\Phi(z;\chi,q)$ we obtain that

$$\Phi(z;\chi,q) = \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{c-i\infty}^{c+i\infty}\Gamma(w)E\Bigg(w;\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Bigg)z^{-w}\,dw. \qquad (5)$$

We substitute z^{-1} instead of z in (5). This gives

$$\begin{aligned}\Phi\big(z^{-1};\chi,q\big) &= \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{c-i\infty}^{c+i\infty}\Gamma(w)E\Bigg(w;\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Bigg)z^{w}\,dw\\ &= \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{1-c-i\infty}^{1-c+i\infty}\Gamma(1-w)E\Bigg(1-w;\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Bigg)z^{1-w}\,dw.\end{aligned} \qquad (6)$$

Now we apply the functional equation for the Estermann zeta-function (see [2])

$$E\Big(1-w; \frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}}, 0\Big) = \frac{1}{\pi}\Big(\frac{2\pi(q,a+1)}{q}\Big)^{1-2w}\Gamma^2(w)\Big(E\Big(w; \frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}}, 0\Big) + \cos(\pi w)E\Big(w; -\frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}}, 0\Big)\Big)$$

and

$$\Gamma(1-w) = \frac{\pi}{\Gamma(w)\sin\pi w}.$$

Thus, from (6), we get

$$\begin{aligned}
&\Phi(z^{-1};\chi,q) \qquad (7)\\
&= \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{1-c-i\infty}^{1-c+i\infty}\Big(\frac{2\pi(q,a+1)}{q}\Big)^{1-2w}\Gamma(w)\Big\{\sin^{-1}(\pi w)\\
&\quad\times E\Big(w; \frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}}, 0\Big) + \cot(\pi w)E\Big(w; -\frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}}, 0\Big)\Big\}z^{1-w}\,dw\\
&= -\frac{2\pi i\delta z}{q}\Big(\frac{4\pi^2 z}{q^2};\chi,-q\Big) + \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{1-c-i\infty}^{1-c+i\infty}\Big(\frac{2\pi}{q}\Big)^{1-2w}\\
&\quad\times\Gamma(w)\Big\{\sin^{-1}(\pi w)E\Big(w; \frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}}, 0\Big)(q,a+1)^{1-2w} + \cot(\pi w)\\
&\quad\times E\Big(w; -\frac{\overline{(\frac{a+1}{(q,a+1)})}}{\frac{q}{(q,a+1)}}, 0\Big)(q,a+1)^{1-2w} + \delta i E\Big(w; -\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}}, 0\Big)\Big\}z^{1-w}\,dw, \qquad (8)
\end{aligned}$$

since $0<1-c<1$. Now we transform the last integral. From formulae (2)–(4), using the expansions

$$\Big(\frac{2\pi(q,a+1)}{q}\Big)^{1-2w} = \frac{q}{2\pi(q,a+1)} - \frac{q}{\pi(q,a+1)}\log\frac{2\pi(q,a+1)}{q}(w-1) + \frac{q}{\pi(q,a+1)}(\cdots)(w-1)^2 + \cdots,$$

$$\sin^{-1}(\pi w) = -\frac{1}{\pi(w-1)} - \frac{\pi}{6}(w-1) + \cdots$$

and

$$\cot(\pi w) = \frac{1}{\pi(w-1)} - \frac{\pi}{3}(w-1) + \cdots,$$

we find that

$$\operatorname*{Res}_{w=1}\left(\frac{2\pi}{q}\right)^{1-2w}\Gamma(w)\Biggl\{\sin^{-1}(\pi w)E\Biggl(w;\frac{\overline{\bigl(\frac{a+1}{(q,a+1)}\bigr)}}{\frac{q}{(q,a+1)}},0\Biggr)(q,a+1)^{1-2w}$$

$$+\cot(\pi w)E\Biggl(w;-\frac{\overline{\bigl(\frac{a+1}{(q,a+1)}\bigr)}}{\frac{q}{(q,a+1)}},0\Biggr)(q,a+1)^{1-2w}+\delta iE\Biggl(w;-\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Biggr)\Biggr\}z^{1-w}$$

$$=-\frac{1}{4}+\frac{\delta i}{2\pi}(q,a+1)\biggl(\gamma-\log\frac{4\pi^2z}{(q,a+1)^2}\biggr)-\frac{q}{2\pi^2(q,a+1)}\bigl(a_{0a}^{+}-a_{0a}^{-}\bigr).$$

Therefore,

$$\frac{1}{2\pi iG(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{1-c-i\infty}^{1-c+i\infty}\left(\frac{2\pi}{q}\right)^{1-2w}\Gamma(w)\Biggl\{\sin^{-1}(\pi w)E\Biggl(w;\frac{\overline{\bigl(\frac{a+1}{(q,a+1)}\bigr)}}{\frac{q}{(q,a+1)}},0\Biggr)$$

$$\times(q,a+1)^{1-2w}+\cot(\pi w)E\Biggl(w;-\frac{\overline{\bigl(\frac{a+1}{(q,a+1)}\bigr)}}{\frac{q}{(q,a+1)}},0\Biggr)(q,a+1)^{1-2w}$$

$$+\delta iE\Biggl(w;-\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Biggr)\Biggr\}z^{1-w}\,dw$$

$$=\frac{1}{2\pi iG(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{q}\right)^{1-2w}\Gamma(w)$$

$$\times\Biggl\{\sin^{-1}(\pi w)E\Biggl(w;\frac{\overline{\bigl(\frac{a+1}{(q,a+1)}\bigr)}}{\frac{q}{(q,a+1)}},0\Biggr)(q,a+1)^{1-2w}+\cot(\pi w)E\Biggl(w;-\frac{\overline{\bigl(\frac{a+1}{(q,a+1)}\bigr)}}{\frac{q}{(q,a+1)}},0\Biggr)$$

$$\times(q,a+1)^{1-2w}+\delta iE\Biggl(w;-\frac{\frac{a+1}{(q,a+1)}}{\frac{q}{(q,a+1)}},0\Biggr)\Biggr\}z^{1-w}\,dw$$

$$+\frac{1}{4G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)-\frac{\delta i}{2\pi G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a+1)\biggl(\gamma-\log\frac{4\pi^2z}{(q,a+1)^2}\biggr)$$

$$+\frac{q}{2\pi^2G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a+1)}\bigl(a_{0a}^{+}-a_{0a}^{-}\bigr),$$

where $1<b<2$. From this and (7), the theorem follows.

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REZIUMĖ

Transformacijos formulė su primityviuoju charakteriu
A. Balčiūnas

Straipsnyje gauta formulė eksponentinei eilutei su primityviuoju charakteriu. Ši formulė gali būti pritaikyta L-funkcijos su primityviuoju charakteriu kvadrato Melino transformacijos meromorfiniam tęsiniui gauti.

Raktiniai žodžiai: Estermano dzeta-funkcija, Gauso suma, Melino transformacija.