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On large deviations for compound mixed Poisson process

Aurelija Kasparavičiūtė, Dovilė Deltuvienė

Vilnius Gediminas Technical University
Saulėtekio 11, LT-10223, Vilnius
E-mail: aurelija@czv.lt, doviled@mail.lt

Abstract. This paper is designated for normal approximation to the distribution function of the compound mixed Poisson process taking into consideration large deviations both in the Cramér and power Linnik zones.

Keywords: cumulants, large deviations, exponential inequalities, compound mixed Poisson process.

Introduction

Let us consider a special Cox process – mixed Poisson process $N_t := N'_{\Lambda(t)}$, $t > 0$, where the mean value function $\Lambda(t)$ is a general random process with non-decreasing sample paths, independent of the standard Poisson process N' (for more details see, e.g., [5, 7]). Such processes have proved useful, for example, in medical statistics where every sample path represents the medical history of a particular patient which has his/her own mean value function.

By a mixed Poisson distribution with the mixing distribution $F_{\Lambda(t)}(x) = \mathbf{P}(\Lambda(t) < x)$ we will mean (see, e.g., [6, 5])

$$q_s = \mathbf{P}(N_t = s) = \frac{1}{s!}\int_0^\infty e^{-x}x^s\,dF_{\Lambda(t)}(x), \quad s \in \mathbb{N}_0. \tag{1}$$

The most well-known and most widely used mixed Poisson distribution is the negative binomial distribution which is generated by the mixing Gamma distribution. To elaborate on, assume that $\Lambda(t)$ is distributed according to the gamma law with positive parameters (n_t, b_t) and density function

$$p_{\Lambda(t)}(x) = \frac{b_t^{n_t}}{\Gamma(n_t)}x^{n_t-1}e^{-b_t x}, \quad x > 0, \tag{2}$$

where $\Gamma(n_t) = \int_0^\infty x^{n_t-1}e^{-x}\,dx$ is gamma function. Obviously, considering on (1) and (2), N_t is distributed according to the negative binomial law with the following probability q_s and parameters $0 < \bar{p} < 1$, $n_t > 0$:

$$q_s = \frac{\Gamma(n_t+s)}{\Gamma(s+1)\Gamma(n_t)}\bar{p}^{n_t}(1-\bar{p})^s, \quad \bar{p} = \frac{b_t}{1+b_t},\ s \in \mathbb{N}_0. \tag{3}$$

N_t process is called negative binomial or Pólya process and is often used in insurance and other dynamic population models. If n_t is positive integer, then negative binomial distribution is called Pascal distribution, and in case $n_t = 1$ – geometric distribution. In virtue of (2), (3), the mean and the variance of N_t and $\Lambda(t)$ are

$$\mathbf{E}N_t = \alpha_t, \qquad \beta_t^2 = \mathbf{D}N_t = \alpha_t + \hat{\beta}_t^2 = \alpha_t(1 + \alpha_t/n_t) > \alpha_t, \tag{4}$$

where $\mathbf{E}\Lambda(t) = \alpha_t = n_t/b_t$, $\mathbf{D}\Lambda(t) = \hat{\beta}_t^2 = n_t/b_t^2$. The property $\beta_t^2 > \alpha_t$ for any $t > 0$ is well-known and is called over-dispersion.

Suppose that $\{X,\ X_j,\ j = 1, 2, \ldots\}$ is a family of independent identically distributed (i.i.d) random variables (r.vs.) with mean $\mu = \mathbf{E}X$, finite, positive variance $\sigma^2 = \mathbf{D}X < \infty$ and having a distribution function $F_X(x) = \mathbf{P}(X < x)$ for all $x \in \mathbb{R}$. The kth order cumulants and the characteristic function (ch.f.) of the random variable (r.v.) X will be denoted by $\Gamma_k(X)$, $k = 1, 2, \ldots,$ $f_X(u) = \mathbf{E}\exp\{iuX\}$, $u \in \mathbb{R}$, respectively (for definitions see [9, pp. 6–8]). Furthermore, we say that the centered moments of the r.v. X with $\sigma^2 < \infty$ satisfy generalized S.N. Bernstein's condition (Bernstein's condition is placed, e.g., in [9, p. 42]): if there exist $\gamma \geqslant 0$ and $K > 0$ such that

$$\left|\mathbf{E}(X - \mu)^k\right| \leqslant (k!)^{1+\gamma} K^{k-2} \sigma^2, \quad k = 3, 4, \ldots. \tag{$\bar{B}_\gamma$}$$

Taking into consideration that $\Gamma_k(X) = \Gamma_k(X - \mu)$, $k = 2, 3, \ldots$ and according to Lemma 3.1 in [9, p. 42], we take up the position that

$$\left|\Gamma_k(X)\right| \leqslant (k!)^{1+\gamma} M^{k-2} \sigma^2, \quad M = 2\max\{\sigma, K\} := 2(\sigma \vee K),\ k = 3, 4, \ldots. \tag{5}$$

In this paper we consider the compound mixed Poisson process

$$S_{N_t} = \sum_{j=1}^{N_t} X_j, \quad S_0 = 0, \tag{6}$$

where we suppose that N_t for each $t > 0$ is independent of $\{X,\ X_j,\ j = 1, 2, \ldots\}$. Since in most cases the accurate distribution for the sum (6) is not available, derivering asymptotic relationship for it's tail probability becomes important. Such asymptotic results often appear in actuarial situations, see, e.g., [5, 7].

The aim of this paper is to consider an instance of large deviation theorems for a distribution function of a sum of a random number of summands of i.i.d, weighted r.vs., considered in the papers [3, 2]. That is, to obtain large deviation theorems both in the Crámer and power Linnik zones for a distribution function of standardized compound mixed Poisson process

$$\tilde{S}_{N_t} = \frac{S_{N_t} - \mathbf{E}S_{N_t}}{\sqrt{\mathbf{D}S_{N_t}}}, \tag{7}$$

where with reference to (8) in [2, p. 2],

$$\mathbf{E}S_{N_t} = \mu\alpha_t, \qquad \mathbf{D}S_{N_t} = \beta_t^2\mu^2 + \alpha_t\sigma^2. \tag{8}$$

Here α_t, β_t^2 are defined by (4). To achieve the purpose of this paper, the cumulant method that was offered by V. Statulevičius (1966) and generalized by R. Rudzkis, L. Saulis, V. Sataulevičius (1978) (for references see [9]), is used.

There is a very extensive literature on the asymptotic behavior of (6) (see, e.g., [1, 4, 6, 5, 7]). For example, general theorems presenting necessary and sufficient conditions for the convergence of the distributions of (6) with non-zero and zero mean have been proved, e.g., in [6, 5]. In [1] logarithmic asymptotic for probabilities of large deviations for compound Cox processes have been derivered. Large deviation results for generalized compound negative binomial risk models, in case considered i.i.d. r.v. have heavy-tailed distribution function, have been extended and improved in [4]. However, between the huge amount of literature authors have not found any published results on considered problem in case when cumulant method is employed, although it is a powerful method that permits the systematic investigation of large deviations for the distributions of sums of a random number of summands.

1 The upper estimates for the cumulants

Since we are interested in a more accurate asymptotic analysis $F_{\tilde{S}_{N_t}}(x)$, at first we must find the accurate upper estimates for the kth order cumulants $\Gamma_k(\tilde{S}_{N_t})$, $k = 3, 4, \ldots$. Considering the following relation (for more details see [3, p. 136])

$$\Gamma_k(S_{N_t}) = k! \sum_1^* \frac{\Gamma_m(N_t)}{m_1! \cdot \ldots \cdot m_k!} \prod_{j=1}^k \left(\frac{1}{j!} \Gamma_j(X) \right)^{m_j}, \quad k = 1, 2, \ldots, \tag{9}$$

it becomes obvious in order to obtain upper bounds for $\Gamma_k(\tilde{S}_{N_t})$, we must impose conditions not only for the kth order cumulants of the r.v. X but for N_t, too. Here $\sum_1^*$ denotes a summation over all non-negative integer solutions $0 \leqslant m_1, \ldots, m_k \leqslant k$ of the equation $m_1 + 2m_2 + \cdots + km_k = k$, $m_1 + \cdots + m_k = m$, $1 \leqslant m \leqslant k$.

Proposition 1. *Assume that $\Lambda(t) > 0$, $t > 0$, is distributed according to the gamma law with the parameters $n_t > 0$, $0 < b_t \leqslant 1$ and density function (2). Then for the kth order cumulants of the mixed Poisson process N_t the upper estimate (L_t) holds:*

$$\Gamma_k(N_t) \leqslant (k-1)! \frac{n_t}{2} \left(\frac{2}{b_t} \right)^k, \quad k = 1, 2, \ldots. \tag{L_t}$$

Proof of Proposition 1. Pursuant to (2),

$$f_{\Lambda(t)}(u) = \mathbf{E} e^{iu\Lambda(t)} = (1 - iu/b_t)^{-n_t}, \quad u \in \mathbb{R}. \tag{10}$$

Wherefore, the definition of the kth order cumulants leads to

$$\Gamma_k(\Lambda(t)) = \frac{1}{i^k} \frac{d^k}{du^k} \ln f_{\Lambda(t)}(u) \bigg|_{u=0} = (k-1)! n_t / b_t^k, \quad k = 1, 2, \ldots. \tag{11}$$

Further, in view of (3) together with (10) the ch.f. of N_t is

$$f_{N_t}(u) = \big((1 - (1-\bar{p}) e^{iu}) / \bar{p} \big)^{-n_t} = f_{\Lambda(t)} \big(\ln f_{N_1}(u)/i \big), \tag{12}$$

where $0 < \bar{p} < 1$ is defined by (3), and N_1 is distributed according to the Poisson law with unit parameter. It is easy to make sure that $\ln f_{N_1}(u) = \exp\{iu\} - 1$.

Now let us derive the kth order cumulants of N_t. Based on Lemma 1 in [8, p. 135], taking into account the definition of the kth order cumulants together with (12) and $\Gamma_k(N_1) = 1$, $k = 1, 2, \dots$, allows one to obtain

$$\Gamma_k(N_t) = \sum_{l=0}^{k-1} c_{k-l}^{(k)} \Gamma_{k-l}\big(\Lambda(t)\big), \quad k = 1, 2, \dots. \tag{13}$$

Integers $c_j^{(k)} \geqslant 1$, $j = 1, 2, \dots, k$, are Stirling numbers of the second kind that can be determined, e.g., from $c_{k-l}^{(k)} = k! \sum_1^{**} \prod_{j=1}^k (1/j!)^{m_j} (1/m_j!)$, $l = 0, 1, \dots, k-1$, $k = 1, 2, \dots$, where $\sum_1^{**}$ denotes the same summation as in case of $\sum_1^{*}$, supposing $m = k - l$, $0 \leqslant l \leqslant k-1$.

If $0 < b_t \leqslant 1$, then substituting (11) into (13) leads to (L_t), due to $\sum_{l=0}^{k-1} c_{k-l}^{(k)} (k - l - 1)! \leqslant (k-1)! 2^{k-1}$, $k = 1, 2, \dots$.

Proposition 2. *If for the r.vs. X, N_t, $t > 0$, conditions, accordingly, $(\bar{B}_\gamma)$ and (L_t) are fulfilled, then*

$$\big|\Gamma_k(\tilde{S}_{N_t})\big| \leqslant (k!)^{1+\gamma} / \Delta_t^{k-2}, \quad \Delta_t = b_t \sqrt{\mathbf{D} S_{N_t}} / \bar{M}_j, \quad j = 1, 2, \ k = 3, 4, \dots, \tag{14}$$

as $0 < b_t \leqslant 1$, where

$$\bar{M}_1 = 2\big(2|\mu| \vee \big(1 \vee \sigma/\big(2|\mu|\big)\big) M\big) \quad \text{as } \mu \neq 0, \qquad \bar{M}_2 = 2M \, \text{as } \mu = 0. \tag{15}$$

Here $M > 0$, $\mathbf{D} S_{N_t}$ are defined, accordingly, by (5), (8).

Proof of Proposition 2. Let us consider the case when $\mu \neq 0$. The application of (9) together with (5), (L_t) leads to

$$\begin{aligned} \big|\Gamma_k(S_{N_t})\big| \leqslant & (k!)^{1+\gamma} M^{k-2} \alpha_t \sigma^2 + k! n_t \sum\nolimits_2^* \frac{(\tilde{m}-1)!}{m_1! \cdot \dots \cdot m_{k-1}!} \frac{2^{\tilde{m}-1}}{b_t^{\tilde{m}}} \\ & \times |\mu|^{m_1} \prod_{j=2}^{k-1} \big((j!)^\gamma M^{j-2} \sigma^2\big)^{m_j}, \quad k = 2, 3, \dots, \end{aligned} \tag{16}$$

where $\sum_2^*$ denotes a summation over all non-negative integer solutions $0 \leqslant m_1, \dots, m_{k-1} \leqslant k$, of the equation $m_1 + 2m_2 + \dots + (k-1)m_{k-1} = k$, $m_1 + \dots + m_{k-1} = \tilde{m}$. Here $1 \leqslant \tilde{m} \leqslant k$.

Clearly, $n_t / b_t^{\tilde{m}} < \beta_t^2 / b_t^{\tilde{m}-2}$ as, certainly, $\beta_t^2 > \hat{\beta}_t^2$. Here β_t, $\hat{\beta}_t^2$ are defined by (4). Consequently, by $\sum_2^* \tilde{m}! / (m_1! \cdot \dots \cdot m_{k-1}!) = 2^{k-1} - 1$, $|\mu|^{m_1} \prod_{j=2}^{k-1} (M^{j-2} \sigma^2)^{m_j} \leqslant |\mu|^{\tilde{m}} ((1 \vee \sigma/(2|\mu|)) M)^{k-\tilde{m}}$, $\prod_{j=1}^{k-1} (j!)^{m_j} \leqslant (k-1)!$, $k = 2, 3, \dots$ (for more details see (22)–(24) in [2, p. 261]) and inequality (16), we arrive at

$$\big|\Gamma_k(S_{N_t})\big| \leqslant (k!)^{1+\gamma} \mathbf{D} S_{N_t} (\bar{M}_1 / b_t)^{k-2} \quad \text{as } 0 < b_t \leqslant 1, \ k = 2, 3, \dots. \tag{17}$$

Here $\mathbf{D} S_{N_t}$, $\bar{M}_1$ are defined by (8), (15), respectively.

Now let us consider the case when $\mu = 0$ and suppose that $0^0 = 1$. Equality (9) and inequalities (5), (L_t) ensure

$$\big|\Gamma_k(S_{N_t})\big| \leqslant k! \alpha_t \sum\nolimits_3^* \frac{(\bar{m}-1)!}{m_2! \cdot \dots \cdot m_k!} \left(\frac{2}{b_t}\right)^{\bar{m}-1} \prod_{j=2}^k \big((j!)^\gamma \sigma^2 M^{j-2}\big)^{m_j} \tag{18}$$

for $k = 2, 3, \ldots$. Here $\sum_3^*$ denotes a summation over all non-negative integer solutions $0 \leqslant m_2, \ldots, m_k \leqslant k$ of the equation $2m_2 + \cdots + km_k = k$, $m_2 + \cdots + m_k = \bar{m}$, $1 \leqslant \bar{m} \leqslant k$.

In accordance with $\prod_{j=2}^k (M^{j-2}\sigma^2)^{m_j} \leqslant \sigma^{2\bar{m}} M^{k-2\bar{m}}$, $\sum_3^* \bar{m}!/(m_2! \cdot \cdots \cdot m_k!) \leqslant 2^{k-2}$, $\prod_{j=2}^k (j!)^{m_j} \leqslant k!$, $k = 2, 3, \ldots$ (for more details see (21), (22) in [3, p. 140]) the estimate of (18) is

$$\left|\Gamma_k(S_{N_t})\right| \leqslant (k!)^{1+\gamma} \mathbf{D} S_{N_t} (\bar{M}_2/b_t)^{k-2} \quad \text{as } 0 < b_t \leqslant 1,\ k = 2, 3, \ldots. \tag{19}$$

Here $\mathbf{D}S_{N_t}$, $M > 0$ stand with $\mu = 0$, and $\bar{M}_2$ is defined by (15). To complete the proof, it is sufficient to use (17), (19) and by noticing that $\Gamma_k(\tilde{S}_{N_t}) = \Gamma_k(S_{N_t})/(\mathbf{D}S_{N_t})^{k/2}$, we arrive at (14).

2 Large deviation theorems

Since the majorating upper estimate for $\Gamma_k(\tilde{S}_{N_t})$, $k = 3, 4, \ldots$ is derivered, we can assert that theorem on large deviations and exponential inequalities follow directly from Theorems 1, 3 in [3, pp. 134–135] with the parameters $\tilde{Z}_N := \tilde{S}_{N_t}$, $\Delta_* := \Delta_t$, where $\tilde{S}_{N_t}$ and Δ_t are defined, accordingly, by (7), (14).

Set $\Phi(x)$ as the standard normal distribution function and assume that $n_t b_t \to \infty$ as $t \to \infty$.

Theorem 1. *Suppose that the r.vs. X, N_t satisfy conditions $(\bar{B}_\gamma)$ and (L_t), respectively. Then*

$$\frac{1 - F_{\tilde{S}_{N_t}}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{S}_{N_t}}(-x)}{\Phi(-x)} \to 1 \tag{20}$$

in both cases: $\mu \neq 0$, $\mu = 0$, hold for $x \geqslant 0$, $x = o((n_t b_t)^{\nu(\gamma)/2})$ as $t \to \infty$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geqslant 0$.

The proof of Theorem 1. The proof almost immediately follows from Theorem 2 in [3, p. 135], considering the instance when $a_j \equiv 1$, $p = 0$, and $N := N_t$, $t > 0$ is mixed Poisson process with the probability (3).

Following the proof of Theorem 2 it is obvious that we must to show: $\Delta_t \to \infty$ as $t \to \infty$. Recall the definitions of $\mathbf{D}S_{N_t}$, Δ_t by (8), (14), respectively, and let's notice that $\beta_t^2 \geqslant 2\alpha_t$ as $\hat{\beta}_t^2 \geqslant \alpha_t$, $0 < b_t \leqslant 1$. Accordingly, $\Delta_t = b_t\sqrt{\mathbf{D}S_{N_t}}/\bar{M}_j \geqslant C_j (n_t b_t)^{1/2}$, $C_1 = \sqrt{2\mu^2 + \sigma^2}/\bar{M}_1$, $C_2 = \sigma/\bar{M}_2 > 0$. Thus, assuming that $n_t b_t \to \infty$ as $t \to \infty$, we achieve that $\Delta_t \to \infty$.

Theorem 2. *If for considered r.vs. X, N_t, $t > 0$, conditions, accordingly, $(\bar{B}_\gamma)$ and (L_t) are fulfilled, then for all $x \geqslant 0$*

$$\mathbf{P}(\pm\tilde{S}_{N_t} \geqslant x) \leqslant \begin{cases} \exp\{-x^2/4\}, & 0 \leqslant x \leqslant (2^{(1+\gamma)^2}\Delta_t)^{1/(1+2\gamma)}, \\ \exp\{-(x\Delta_t)^{1/(1+\gamma)}/4\}, & x \geqslant (2^{(1+\gamma)^2}\Delta_t)^{1/(1+2\gamma)}. \end{cases}$$

Example 1. Assume that $n_t = n \in \mathbb{N}$ and b_t is fixed. Then relations (20) hold for $x = o(n^{\nu(\gamma)/2})$ as $n \to \infty$.

Example 2. If $b_t = 1/t \leqslant 1$, and parameters n_t, $b_t > 0$ of gamma distribution are related by $n_t = (nt^{2(1-\epsilon)})^{1/\epsilon}$, $n \in \mathbb{N}$, where $0 < \epsilon \leqslant 1$, then it can be proved that Lemma 1 (on regularity condition for the kth order cumulants of the sum of the random number of summands of i.i.d. weighted r.vs.) in [3, p. 131] in case $\mu \neq 0$ holds with $N := N_t$, $a_j \equiv 1$, $j = 1, 2, \ldots$, $K_1 := 2/\sqrt{n}$, $p := \epsilon/2$. Suppose that n is fixed. Thus in this instance relations (20) hold for $x \geqslant 0$ such that $x = o(t^{\nu(\gamma)(2-3\epsilon)/(2\epsilon)})$, $0 < \epsilon \leqslant 2/3$ as $t \to \infty$. Assuming that parameters n_t, b_t are related by $n_t = (nt^{1-\bar{\epsilon}})^{1/\bar{\epsilon}}$, $0 < \bar{\epsilon} \leqslant 1$, we have that Lemma 1 in [3, p. 131] in case $\mu = 0$ stands with $N := N_t$, $a_j \equiv 1$, $j = 1, 2, \ldots$, $K_2 := 2/n$, $p := \bar{\epsilon}$. Then (20) hold for $x = o(t^{\nu(\gamma)(1-2\bar{\epsilon})/(2\bar{\epsilon})})$, $0 < \bar{\epsilon} < 1/2$ as $t \to \infty$.

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REZIUMĖ

Didieji nuokrypiai mišriam sudėtiniam Puasono procesui

A. Kasparavičiūtė, D. Deltuvienė

Šiame darbe yra nagrinėjama sudėtinio mišraus Puasono proceso normalioji aproksimacija didžiųjų nuokrypių Kramero ir laipsninėse Liniko zonose, taikant kumuliantų metodą.

Raktiniai žodžiai: kumuliantai, didieji nuokrypiai, sudėtinis mišrus Puasono procesas, eksponentinės nelygybės.