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Modeling of surfaces using 3D fractal interpolation

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Abstract. This paper discusses the concept of three-dimensional (3D) fractal interpolation and the possibility to use it in modeling 3D surfaces. It is important to notice that this paper treats fractal interpolation only as a numerical tool and not as a model. The purpose of the research is to create a methodology for obtaining models for the given 3D surface and making them similar to it to a certain degree. The set of models then can be investigated as required (3D graphical representation, simulation of particular technological process, quality assessment for bonded surfaces, etc.). Measuring a particular 3D surface and making a set of models is far more cost efficient than performing the measurements many times.

Keywords: Fractal interpolation, three-dimensional surface, model.

Introduction

Previous researches [1] showed the possibility to use fractal interpolation (FI) as a tool for simulating rough (locally ragged and irregular) surfaces. Although FI based technique was applied to the profilogram and the resulting curve was extruded to 3D. This paper takes one step further and introduces FI based technique for simulating 3D surface directly in 3D space. And this step results in a whole new quality of the model.

It is important to mention that FI has an important quality not to average the data between the interpolation points compared to other interpolation schemes [2]. In fact this makes it more versatile.

1 Fractal interpolation

The idea of fractal interpolation is to compute the union of the copies of the initial set B_0 than take that union as B_0 and repeat the process indefinitely. Resulting surface is a fractal. B_0 is chosen arbitrary but must cover the same domain on the xOy plane as the surface modeled. This requirement also applies to the union of copies. The affine transformation (the one which result is bounded) in 3D space used in FI is written as (1).

$$\omega_i\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}+\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}x^\star\\y^\star\\z^\star\end{bmatrix},\quad i=\overline{1,n}. \tag{1}$$

Coefficients a_{ij}, $i,j \in \{1;2;3\}$, and b_i, $i \in \{1;2;3\}$, represent a unique transformation. There is a total number of n transformations. b_1, b_2 and b_3 controls shifting along the axis Ox, Oy and Oz accordingly.

Taking in mind that it is only required to shift, scale or shear along the Ox or the Oy axis the copies on B_0 and then put them all into union, the share transformations along the Oz axis are not required. So the parameters a_{12}, a_{13} and a_{23} are set to 0. There are now 9 unknowns in each expression of ω_i, $i = \overline{1,n}$. By writing an equality (1) three equations are produced. So three points are needed to obtain 9 equations.

$$\begin{cases} \omega_i\big([x_1 \quad y_1 \quad z_1]^T\big) = \big[x_1^\star \quad y_1^\star \quad z_1^\star\big]^T, \\ \omega_i\big([x_2 \quad y_2 \quad z_2]^T\big) = \big[x_2^\star \quad y_2^\star \quad z_2^\star\big]^T, \\ \omega_i\big([x_3 \quad y_3 \quad z_3]^T\big) = \big[x_3^\star \quad y_3^\star \quad z_3^\star\big]^T, \quad i = \overline{1,n}. \end{cases} \tag{2}$$

Having Eqs. (2) the unique set of unknown parameters for each ω_i, $i = \overline{1,n}$, can be found. This is due to the fact that there are the same number of linear equations and unknown parameters. The algebraic form of linear equations (2) for any ω_i is:

$$\begin{cases} a_{11}x_1 + b_1 = x_1^\star, \\ a_{21}x_1 + a_{22}y_1 + b_2 = y_1^\star, \\ a_{31}x_1 + a_{32}y_1 + a_{33}z_1 + b_3 = z_1^\star, \\ a_{11}x_2 + b_1 = x_2^\star, \\ a_{21}x_2 + a_{22}y_2 + b_2 = y_2^\star, \\ a_{31}x_2 + a_{32}y_2 + a_{33}z_2 + b_3 = z_2^\star, \\ a_{11}x_3 + b_1 = x_3^\star, \\ a_{21}x_3 + a_{22}y_3 + b_2 = y_3^\star, \\ a_{31}x_3 + a_{32}y_3 + a_{33}z_3 + b_3 = z_3^\star. \end{cases}$$

2 The model of a surface

The parameter a_{33} for each of the transformations are thought to be the control parameter of a surface model. The model presented in this paper is rather simple. The original surface dictates the global shape while the control parameter adjusts the smoothness of the model.

In order to join the copies of B_0 smoothly it is required that B_0 must be equal to original surface function at marginal coordinates. Just for the purpose of better understanding the problem and visual its representation B_0 is chosen to be of a shape of paraboloid. The paraboloid can be easily determined over a certain domain in the plane. The problem here is that the limiting curves between the 3 points which are used for determining particular affine transformations must be straight lines. This requirement is considered to avoid situations when two residing side by side copies of B_0 comes together in different heights. Summing everything up results in finding a paraboloid like shape surface having its base flat on a triangle. Fig. 1 illustrates surface for possible use as B_0. Such surface can be obtained as a product of 3 extruded (along certain axis) parabolas (let us call them by only parabolas).

Fig. 2 shows parabolas extruded along 3 different axes. The product of these parabolas is the shape in Fig. 1. If smooth model without any ragged edges is required

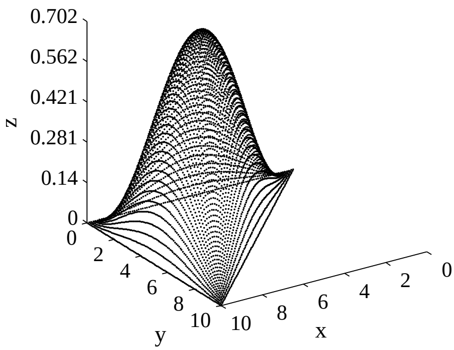
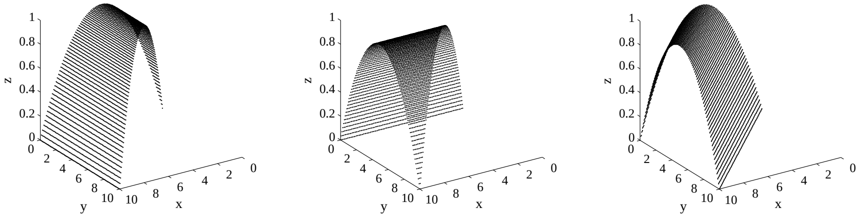


Fig. 1. Initial set B_0 (paraboloid shaped and having its base flat on a triangle).



(a) $z_1(x, y)$. (b) $z_2(x, y)$. (c) $z_3(x, y)$.

Fig. 2. Extruded parabolas.

drawback arises. The copies of such B_0 may (and in most cases will) come together at an angle. This fact is the object of future research.

$$z_1(x, y) = 1 - \frac{(x - \frac{a}{2})^2}{0.25 \cdot a^2}, \tag{3}$$

$$z_2(x, y) = 1 - \frac{(y - \frac{a}{2})^2}{0.25 \cdot a^2}, \tag{4}$$

$$z_3(x, y) = 1 - \frac{(\frac{x\sqrt{2}}{2} - \frac{y\sqrt{2}}{2} - \frac{a\sqrt{2}}{4})^2}{0.125 \cdot a^2}. \tag{5}$$

During the research Eqs. (3), (4) and (5) were formed. Here the parameter a denotes the height of the extruded parabolas. a is arbitrary and has the same effect to the model as previously discussed a_{33}. The process of deriving these equations contained the rotation of the coordinate system and the transformation of the plot of a parabola. The plots of these equations are depicted in Fig. 2. The initial set B_0 is the surface $z(x, y) = z_1(x, y) \cdot z_2(x, y) \cdot z_3(x, y)$.

3 Example of a surface model

Consider the simple paraboloid shaped surface (Fig. 3) to be modeled. Let this surface be denoted as S_{orig}.

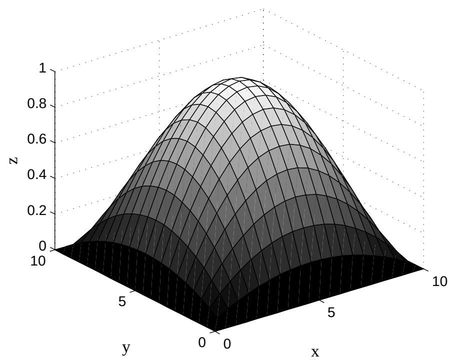


Fig. 3. Example surface to be modeled.

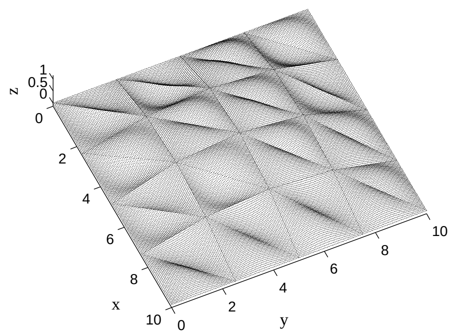


Fig. 4. The regions of the plane xOy for copies of B_0.

Firstly the division into triangles of the area below S_{orig} must be performed. The view of this process is depicted in Fig. 4. The second step is to find all affine transformations for B_0 to be transformed into the location of 3D space over these triangles. Often an element of randomness is required in the model. Here each affine transformation could contain random parameter a_{33} resulting in random heights of the copies of B_0. Of course one can set $a_{33} = a_{33} + cost$ in order to force particular height over a certain region in xOy.

Fig. 5 shows first iteration of fractal interpolation. This surface can be treated as the model for S_{orig}. The model is just approximation of the fractal which will be constructed if infinite number of iterations would be performed. That is why FI acts only as a tool for the model. Nevertheless there are many more things to discuss. For example the problem of how the copies of B_0 come together when performing the second iteration of FI.

4 Concluding remarks

1. Research showed that 3D fractal interpolation can be used as a tool for model 3D surfaces. The roughness of a surface can be changed by adjusting parameter

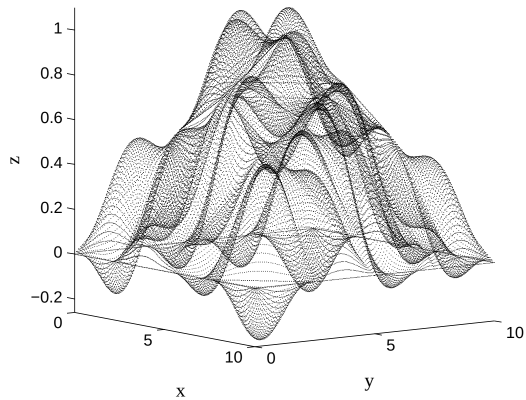


Fig. 5. 3D model of the surface depicted in Fig. 3 as the first iteration of FI.

a_{33} or by performing one step of FI more. Furthermore different values of a_{33} can be used in each iterations to add more variety to the shape of the model.

2. The division of the domain of S_{orig} may not necessary be uniform. Regions may depend on surface roughness, be produced by performing Delaunay triangulation or even chosen randomly.
3. Most of the ideas written in this paper are in developing stage. Future researches will not only improve the model but also consider more tools for making it similar to S_{orig} at a desirable level.

References

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REZIUMĖ

Trimatės fraktalinės interpoliacijos panaudojimas paviršių modeliavimui
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Kai kuriuos realaus pasaulio objektus labai sunku aprašyti. Šio darbo tikslas yra sukurti šiurkštaus paviršiaus modelį, panaudojant fraktalinės geometrijos elementus. Modelis turi iki norimo detalumo atkartoti originalų paviršių. Reikia pabrėžti, kad čia neieškomos fraktalo savybės realiuose duomenyse. Naudojamos fraktalinės geometrijos teikiamos galimybės, kurių neturi Euklido geometrija. Gauti modeliai gali būti naudojami grafiniam paviršių vaizdavimui, technologinio proceso imitavimui, paviršių sanklijų kokybės tyrimams ir kt. Vieną kartą išmatuoti tam tikrą paviršių ir sukurti keletą jo modelių yra greičiau ir pigiau negu matuoti kelis paviršius daug kartų.

Raktiniai žodžiai: Fraktalinė interpoliacija, trimatis paviršius, modelis.