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On some cardinal invariants of hyperspace with (C_0, C_1) Ochan type topology

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Abstract. In the article are examined some cardinal invariants of hyperspace with Ochan type topology.

Keywords: hyperspace, cardinal invariants, Ochan type topology.

1 Introduction

By $\mathcal{P}^*(X)$ we denote the space of all closed subsets of topological space X (or simply of some set X). By $C_0(X)$ or C_0 we note family of all finite subsets of X and $C_1(X)$ or C_1 family of all closed subsets of space X. For family $\mathcal{M} \subset \mathcal{P}^*(X)$ we provide (C_0, C_1) topology by giving base $\{[A, B]_{\mathcal{M}}\colon A \in C_0,\ B \in C_1\}$ there $[A, B]_{\mathcal{M}} = \{(P) \in \mathcal{M}\colon P \subset A \text{ and } P \cap B = \varnothing\}$. If $\mathcal{M}$ is cleare from context we simply note $[A, B]$. $\exp X$ denote hyperspace of space X. All topological notions and terms wich we use can be found in [2]. The first time Ochan type topology was mentioned in [4] and later examined in generale by R. Kašuba [3]. All definitions of cardinal invariants and their basic properties are in [1]. Also all spaces we assumed be Hausdorff.

Now let discus some cardinal invariants starting by.

Lemma 1. *Character of the point (F) in $\exp X$ not exceed τ (there τ some infinite cardinal number) if and only if character $\chi(F, X)$ of subset F in X don't exceed τ and $|F| \leqslant \tau$.*

Proof. Let $\mathcal{U} = \{[A_\alpha, B_\alpha]\colon \alpha \in A\}$ base of space $\exp X$ at the point (F). We'll prove, that $\widetilde{\mathcal{U}} = \{\mathcal{U}_\alpha\colon \alpha \in A\}$, where $\mathcal{U}_\alpha = X \setminus B_\alpha$ for each $\alpha \in A$ is base of subset F in space X. For each neighborhood OF of set F, open set (in $\exp X$) $[\varnothing, X \setminus OF]$ is neighborhood of (F) at $\exp X$. We can find $\alpha^* \in A$ and $[A_{\alpha^*}, B_{\alpha^*}]$, that $(F) \in [A_{\alpha^*}, B_{\alpha^*}] \subset [\varnothing, X \setminus OF]$ and consequently $F \subset \mathcal{U}_{\alpha^*} \subset OF$ and it means that $\widetilde{\mathcal{U}}$ is base of F at X. Because $|\widetilde{\mathcal{U}}| \leqslant |\mathcal{U}|$ than in the case of fulfilment assumption $|\mathcal{U}| \leqslant \tau$ we conclude $\chi(F, X) \leqslant \tau$.

Let $D \subset F$ and $|D| < \aleph_0$. Than open set $[D, \varnothing]$ is neighborhood of (F) at $\exp X$. We can find $\beta \in A$, that $[A_\beta, B_\beta] \subset [D, \varnothing]$ and consequently $D \subset A_\beta$, but than $\bigcup\{A_\alpha\colon \alpha \in A\} = F$ and $|F| \leqslant \tau$ if $|\mathcal{U}| \leqslant \tau$, because A_α are finite for each $\alpha \in A$.

Let prove sufficiency of conditions. Let $\widetilde{\mathcal{U}} = \{U_\alpha\colon \alpha \in A\}$ some base of set F at X. Then $\mathcal{U} = \{[P, X \setminus U_\alpha]\colon P \in C_0(F),\ \alpha \in A\}$ base of (F) at $\exp X$ and $|\mathcal{U}| \leqslant |F| \cdot |\widetilde{\mathcal{U}}|$, consequently $|U| \leqslant \tau \cdot \tau = \tau$ if $|F| \leqslant \tau$ and $|\widetilde{\mathcal{U}}| \leqslant \tau$. The proof is over.

Corollary 1. *Lemma* 1 *allows us to conclude for wich spaces* $\exp X$ *is* I *countable space.*

Proposition 1. *Let space X is Hausdorft. Space* $\exp X$ *is* I *countable space. if and only if when*:

(i) $|X| = \aleph_0$,

(ii) $\mathcal{K} = X \setminus I(X)$ – *compact, where $I(X)$ set of isolated points of space X,*

(iii) *character $\chi(\mathcal{K}, X)$ of set $\mathcal{K}$ in X is countable.*

Proof. Let space $\exp X$ is I countable. Then conditions (i) and (iii) follow from Lemma 1. Prove, that $\mathcal{K} = X \setminus I(X)$ is compact. Because X is countable is sufficient to prove that $\mathcal{K}$ is countable-compact. Let $\mathcal{K}$ is not countably compact, than in $\mathcal{K}$ exist discrete closed subset $A = \{x_i\colon i \in \mathbb{N}\}$, $|A| = \aleph_0$. Let $\mathcal{V} = \{V_i\colon i \in \mathbb{N}\}$ discrete set of neighborhood of point x_i and $x_i \in V_i$ for each $i \in \mathbb{N}$. Let $\{O_i\colon i \in \mathbb{N}\}$ base of neighborhood of set A in X. Choose $y_i \in O_i \cap (V_i \setminus A)$, what is possible, because A haven't isolated points of space X. Than set $B = \{y_i\colon i \in \mathbb{N}\}$ is closed (because $\mathcal{V}$ is discrete), $B \cap A = \varnothing$ and $B \cap O_i \neq \varnothing$ for each $i \in \mathbb{N}$, what contradicts, that $\{O_i\colon i \in \mathbb{N}\}$ is base of neighborhoods of A in X.

Let prove sufficiency of conditions (i), (ii), (iii). For that, using Lemma 1, is sufficient to show that character of each closed set F in X is countable. Let form countable base of F in X. Let $\{ci\colon i \in \mathbb{N}\}$ base of neighborhoods $\mathcal{K}$ in X such that $OK_{i+1} \subset OK_i$ for each $i \in \mathbb{N}$, and $\alpha\beta = \{c_j\colon j \in \mathbb{N}\}$ is sequence of open subsets in X such that $\bigcap\{C_j\colon j \in \mathbb{N}\} = F$ and $C_{p+1} \subset \bar{C}_{p+1} \subset C_p \subset \bar{C}_p \subset OK_p$ for each $p \in \mathbb{N}$. Than $\alpha\beta$ is countable base of neighborhoods of F in X. Proof is over.

Proposition 2. *Let X is normal space. Then tightness $t(\exp X)$ of space* $\exp X$ *is equal to character $\chi(\exp X)$ of space X.*

Proof. Tightness not exceed character of space for each space. So is sufficient to prove that $t(\exp X) \geqslant \chi(\exp X)$. Using Lemma 1 at first we prove $t((F), \exp X) \geqslant |F|$ for each $(F) \in \exp X$. Choise $(F) \in \exp X$ $(F) \in \overline{\{(A)\colon A \subset F,\ |A| < \aleph_0\}}$ (closure is taken in space $\exp X$). But if $(F) \in \bar{N}$, where N some subset of $\{(A)\colon A \subset F,\ |A| < \aleph_0\}$ than $F = \bigcup\{A\colon (A) \in N\}$. Consequently $|F| \leqslant |N|$ and $t((F), \exp X) \geqslant |F|$. Let now $\mathcal{V} = \{(\bar{V})\colon F \subset V,\ IntV = V\}$. Because X is normal $(F) \in \bar{\mathcal{V}}$. On the other hand from $(F) \in \bar{\mathcal{V}}_0$ where $\mathcal{V}_0 \subset \mathcal{V}$ follow $\{IntB\colon (B) \in \mathcal{V}_0\}$ is base of subset F in X. Consequently $t((F), \exp X) \geqslant \chi(F, X)$. Using Lemma 1 we conclude $t((F), \exp X) \geqslant \chi((F), \exp X)$ for each $(F) \in \exp X$. Proof is over.

Corollary 2. *In Proposition* 2 *is proved that tightness and character conside at any point of hyperspace.*

Lemma 2. *Pseudocharacter $\Psi((F), \exp X)$ of point (F) do not excide τ, where τ some cardinal number if and only if, when at the same time*:

(i) *pseudocharacter $\Psi(F, X)$ of F in X do not exceed τ,*

(ii) *density $d(F)$ of set F do not exceed τ.*

Proof. Let $\Psi((F), \exp X) \leqslant \tau$ for some point $(F) \in \exp X$. We can finde family of basic subsets $\mathcal{U} = \{[A_\alpha, B_\alpha]\colon \alpha \in I\}$ that $|\mathcal{U}| \leqslant \tau$ and $(F) = \bigcap\{[A_\alpha, B_\alpha]\colon \alpha \in I\}$.

But $\bigcap\{[A_\alpha, B_\alpha]\colon \alpha \in I\} = [\bigcup\{A_\alpha\colon \alpha \in I\}, \bigcup\{B_\alpha\colon \alpha \in I\}]$. According Lemma 1 $\bigcup\{A_\alpha\colon \alpha \in I\} = F \cap \{X \setminus B_\alpha\colon \alpha \in I\} = F$. Thus A_α, $\alpha \in I$ are finite we can conclude $d(F) \leqslant \tau$. Inequation $\Psi(F, X) \leqslant \tau$ follows from $\bigcap\{X \setminus B_\alpha\colon \alpha \in I\} = F$.

Now let prove sufficiency of conditions (i) and (ii). Let conditions (i), (ii) are satisfied for some $(F) \in \exp X$. Than we prove, that $\Psi((F), \exp X) \leqslant \tau$. For that pick $A \subset F$, that $\bar{A} = F$ and $|A| \leqslant \tau$, and family of neighborhoods $\{O_\alpha\colon \alpha \in I\}$ of set F power wich do not exceed τ, that $\bigcap\{O_\alpha : \alpha \in I\} = F$. Than family of basec subsets $\{[c,\ X \setminus O_\alpha]\colon \alpha \in I,\ c \in C_0(A)\}$ is pseudobasis of (F) in $\exp X$ and it's power do not exceed $\tau \cdot \tau = \tau$. Proof is over. □

Corollary 3. *Pseudocharacter $\Psi(\exp X)$ do not exceed network weight $n\omega(X)$ of regular space X.*

Corollary 4. *Pseudocharacter $\Psi(\exp X)$ is countable if and only if X is perfect and all closed subsets X are separabile.*

Proposition 3. *Pseudoweight $p\omega(\exp X)$ of hyperspace of regular space X don't exceed weight $\omega(X)$ of space X.*

Proof. Let $\mathcal{B}$ is base of the space X power wich is $\omega(X)$. $\widetilde{\mathcal{B}} = \{\bar{V}\colon V \in \mathcal{B}\} \cup \{\varnothing\}$, $\mathcal{PB} = \{\bigcup\{[y, B]\colon y \in V\}\colon B \in \widetilde{\mathcal{B}},\ V \in \mathcal{B}\}$ is pseudobasis of the space $\exp X$ wich power don't exceed $|\mathcal{B}|$. Let discuss $\mathcal{PB}$. $\mathcal{PB}$ really is pseudobasis of $\exp X$. Let $(F) \in \exp X$. We need to prove $\bigcap\{\mathcal{U}\colon \mathcal{U} \in \mathcal{PB},\ (F) \in \mathcal{U}\} = (F)$. Note, that $(P) \in \mathcal{U}$, where $\mathcal{U} \in \mathcal{PB}$ if and only if $P \cap B = \varnothing$ and $P \cap V \neq \varnothing$ for some $B \in \widetilde{\mathcal{B}}$ and $V \in \mathcal{B}$. Let P is closed and $P \neq F$. If $P \cap (X \setminus F) \neq \varnothing$, than because X is regular we can find $W \in \mathcal{B}$ and $D = \bar{W} \in \widetilde{\mathcal{B}}$, that $P \cap D \neq \varnothing$, and $D \cap F = \varnothing$. Let $V_0 \in \mathcal{B}$ and $V_0 \cap F \neq \varnothing$ and $V_0 \cap D = \varnothing$ than $(P) \notin \mathcal{U}_0 \in \mathcal{PB}$, but $(F) \in \mathcal{U}_0$. If $P \subsetneqq F$, we can find $V_1 \in \mathcal{B}$, that $P \cap V_1 = \varnothing$, and $V_1 \cap F \neq \varnothing$. Assume $\mathcal{U}_1 = \bigcup\{[y, \varnothing]\colon y \in V_1\}$. Than $\mathcal{U}_1 \in \mathcal{PB}$, $(F) \in \mathcal{U}_1$, but $(P) \notin \mathcal{U}_1$. Consiquently $\mathcal{PB}$ is pseudobasis of $\exp X$. Let $|\mathcal{B}| = \omega(X)$ than we get $|\mathcal{PB}| \leqslant \omega(X)$. Proof is over. □

Corollary 5. *Let X is regular space and $\mathcal{K} \subset \exp X$ is compact. Than $\omega(\mathcal{K}) \leqslant \omega(X)$.*

Remark 1. Evaluation of Proposition 3 to improve is not possible. Let Y is one point compactification of discrete space power τ. Then $\exp Y$ include Cantor discontinuum $\mathcal{D}^\tau$ weight τ consequently $p\omega(\exp Y) = \omega(Y)$.

Proposition 4. *π – weight of hyperspace $\pi\omega(\exp X)$ is equical* $\max\{|X|, \omega(X)\}$.

Proof. Let $\Pi = \{[A_\alpha, B_\alpha]\colon \alpha \in A\}$ some π − base of hyperspace $\exp X$. Than for each point $x \in X$ and it's neighborhood V we can find $\alpha^* \in A$, that $[A_{\alpha^*}, B_{\alpha^*}] \subset [\{x\},\ X \setminus V]$. Concicvently $x \in A_{\alpha^*} \subset X \setminus B_\alpha \subset V$ and $\mathcal{B} = \{X \setminus B_\alpha\colon \alpha \in A\}$ is base of the space X and $|\mathcal{B}| \leqslant |\Pi|$. Because family $\{A_\alpha\colon \alpha \in A\}$ cover X, and A_α are finite, than $|\Pi| \geqslant |X|$. So we proved, that $\pi\omega(\exp X) \geqslant \max\{\omega(X), |X|\}$.

Now let prove inverse $\pi\omega(\exp X) \geqslant \max\{\omega(X), |X|\}$. Let $\mathcal{B} = \{V_\alpha\colon \alpha \in A\}$ is base of X and $|\mathcal{B}| = \omega(X)$. Than family $\{[P,\ X \setminus V]\colon P \in C_0(X),\ V = \bigcup \mathcal{V} \subset \mathcal{B},\ |\mathcal{V}| < \aleph_0\}$ is π − base for hyperspace $\exp X$ and its power don't exceed $|X| \cdot \omega(X)$. Proof is over.

Theorem 1. *Let X is regular topological space. Density of hyperspace $d(\exp X)$ with (C_0, C_1) topology is equal to network weight $n\omega(X)$ of the space X.*

Proof. Let $\mathcal{N}$ is network of the space X such, that $|\mathcal{N}| = n\omega(X)$. Assume that $\mathcal{N}$ consists of closed X subsets. We prove, that $\widetilde{\mathcal{N}} = \{(\bigcup \mathcal{N}^*)\colon \mathcal{N}^* \subset \mathcal{N},\ |\mathcal{N}^*| < \aleph_0\}$ is close in the space $\exp X$. Let $[A, B]$ is set of standart base. Thus A is finite we can find $\mathcal{N}^{**} \subset \mathcal{N}$, that $A \subset \bigcup \mathcal{N}^{**} \subset X \setminus B$. Consequently $(\bigcup \mathcal{N}^{**}) \in [A, B]$. Thus $[A, B]$ is chose freelly we conclude $\widetilde{\mathcal{N}}$ is network of $\exp X$. Conclude $n\omega(X) \geqslant d(\exp X)$.

Now let prove inverse $d(\exp X) \geqslant n\omega(X)$. Let $\mathcal{M}$ is dense subset of the space $\exp X$ and $|\mathcal{M}| = d(\exp X)$. For each point $x \in X$ and each neighborhood V we can find $(F) \in \mathcal{M}$, such that $(F) \in [\{x\}, X \setminus V]$, consequently $x \in F \subset V$. Because x is freelly chose we conclude that $\widetilde{\mathcal{M}} = F\colon (F) \in \mathcal{M}$ is netwok of the space X and $|\bar{\mathcal{M}}| = |\mathcal{M}|$. Consequently $n\omega(X) \leqslant d(\exp X)$. Proof is over.

Corollary 6. *Pseudocharacter of hyperspace of regular space X don't exceed it's density.*

Theorem 2. *Let X is Hausdorft topological space and it's hyperspace* $\exp X$ *with (C_0, C_1) topology. Then following are equivalent:*

(i) $\exp X$ *is metrizable,*

(ii) *weight of* $\exp X$ *is countable,*

(ii) $\exp X$ *is simetrizable,*

(ii) *spread* $\exp X$ *is countable,*

(iv) $\exp X$ *is hereditary separable,*

(v) X *is countable and compact.*

Proof. At first we note, that from conditions (i), (ii), (iii), (v) follow countability of $\underline{\text{tightness of the space}}$ $\exp X$, but than X countable, because $(X) \in \overline{\{(A)\colon A \subset X,\ |A| < \aleph_0\}}$. From condition (iv) follows countability of X, because $\{(\{x\})\colon x \in X\}$ is discrete subset of hyperspace $\exp X$. So to prove that (vi) follows from each of condition (i), (ii), (iii), (iv), (v) is sufficient to prove that in the case of satisfaction at lest of one of condition (i), (ii), (iii), (iv), (v) space X is countable compact. Assume that from condition (i), (ii), (iii), (iv), (v) don't follow countable compactness of X. Than X contens countable, closed discrete subset D and $\exp D$ is embeddible into $\exp X$ an $\exp D$ topology concide with Vietoris topology. But $\exp D$ is not simmetrizable space and contents discrete subset power $2^{\aleph_0}$. We get contradiction.

Because conditions (i), (iii), (iv), (v) follow from condition (ii) to conclude the proof is sufficient to prove that from (iv) follows $\overline{\text{(ii)}}$. Let $\mathcal{B}$ is countable base of countable and compact space X. Familly $\mathcal{B} = \{\bigcup \mathcal{B}^*,\ |\mathcal{B}^*| < \aleph_0\}$ is also countable, consequently countable is and family $\{[A, B]\colon A \in C_0(X),\ B \in \widetilde{\mathcal{B}}\}$ from base of $\exp X$, because X is compact. Proof is over.

The large role in evaluation of cardinal invariants of $\exp X$ plays power of the space X:

Proposition 5. *Let cardinal number τ_1 is equal to minimal number of $\omega(\exp X)$, $n\omega(\exp X)$, $S(\exp X)$, $l(\exp X)$, $\chi(\exp X)$ and cardinal number τ_2 is equal to largest of numbers $\psi(\exp X)$, $p\omega(\exp X)$, $d(\exp X)$, $c(\exp X)$. Than $\tau_1 \geqslant |X| \geqslant \tau_2$.*

Proof. $\min\{\omega(\exp X), n\omega(\exp X), l(\exp X), S(\exp X)\} \geqslant |X|$ because $\exp X$ contens closed discrete subset $\{(\{x\})\colon x \in X\}$ and it's power $|X|$. Inequation $\chi(\exp X) \geqslant |X|$ follows from Lemma 1. Inequation $|X| \geqslant d(\exp X)$ follows from Teorem 1. Because Suslin's number don't exceed density of the space we get $c(\exp X) \leqslant |X|$; $|X| \geqslant p\omega(\exp X)$, because $\{[\{x\},\{y\}]\colon x, y \in X\}$ pseudobasis of $\exp X$, with power $|X|$. Proof is over.

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REZIUMĖ

Apie kai kurios hypererdvės su Očaniškąja (C_0, C_1) topologija kardinalinius invariantus

G. Praninskas

Straipsnyje nagrinėjami topologinės erdvės uždarų poaibių erdvės (hypererdvės) su (C_0, C_1) Očano tipo topologija kardinaliniai invariantai. Jų ryšys tarpusavyje, o taip pat santykiai tarp topologinės erdvės ir jos hypererdvės kardinalių invariantų.

Raktiniai žodžiai: hypererdvė, Očano tipo topologija, kardinalinis invariantas.