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Method of marks for propositional linear temporal logic

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Abstract. It is known that traditional techniques used to ensure termination of a decision procedure in non-classical logics are based on loop-checking, in general. Nowadays, effective loop-check techniques based on histories are used instead of unrestricted loop-check. These techniques are widely and successfully applied also for non-classical logics containing induction-like axioms. These induction-like axioms create new type loops ("good loops") along with ordinary "bad loops". In this paper, some loop-check free saturation-like decision procedure based on some technique of marks is proposed. This saturation procedure terminates when special type marked sequents are obtained. This procedure is demonstrated for propositional linear temporal logic (**PLTL**) with temporal operators "next" and "always".

Keywords: sequent calculus, loop-check, temporal logics, termination, marks.

Introduction

Temporal logic is a special type of modal logic (see, e.g., [1, 2]). It provides a formal system for qualitatively describing and reasoning about how the truth values of assertions change over time.

It is well known that check of termination of a decision procedure plays a crucial role in constructing derivations. Along with non-classical logics without induction like tools, there are very important for computer science and artificial intelligence non-classical logics containing induction like tools, e.g., temporal, dynamic, common knowledge logics and their various modifications and combinations. Usually, these induction like tools are realized using loop-type axioms. Determination of these loop-type axioms involves creating new loops ("good loops" in opposite to "bad loops") and the new loop checking along with ordinary non-induction-type loop checking (see, e.g., [3]).

In the present paper, a propositional linear temporal logic with temporal operators $\circ$ ("next") and $\Box$ ("always") is considered. It is known that combination of these temporal operators requires to use induction-like tools. To determine such tools, some simple saturation procedure, based on the technique of marks, is proposed. This procedure allows us to eliminate the search of "bad" and "good" loops at all. Instead of these loops, the proposed procedure generates some marked sequents of special shape. The constructed procedure is loop-check-free and backtracking-free.

The paper is organized as follows. In Section 1, initial loop-type procedure for **PLTL** is described. In Section 2, a proposed loop-check-free backward proof-search procedure is described. Some examples demonstrating the procedure are presented. Foundation of the procedure is proved in Section 3.

1 Initial loop-type calculus for PLTL

The language of considered **PLTL** contains a set of propositional symbols $P, P_1, P_2, \ldots, Q, Q_1, Q_2, \ldots$; the set of logical connectives $\supset, \wedge, \vee, \neg$; temporal operators $\Box$ ("always") and $\circ$ ("next"). The language does not contain the temporal operator $\diamond$ ("sometimes"), assuming that $\diamond A = \neg\Box\neg A$. We assume that time is linear, discrete, and ranges over the set of natural numbers.

Formulas in the considered calculi are constructed in the traditional way from propositional symbols, using the logical connectives and temporal operators. The formula $\circ A$ means "A is true at the next moment of time"; the formula $\Box A$ means "A is true now and in all moments of time in the future".

We consider sequents, i.e., formal expressions $\Gamma \to \Delta$, where Γ and Δ are finite sets of formulas.

A sequent S is a *primary* (*quasi-primary*) one, iff $S = \Sigma_1, \circ\Gamma_1 \to \circ\Gamma_2, \Sigma_2$ ($S = \Sigma_1, \circ\Gamma_1, \Box\Delta_1 \to \circ\Gamma_2, \Box\Delta_2, \Sigma_2$), where Σ_i ($i \in \{1,2\}$) is empty or consists of propositional symbols; $\circ\Gamma_i$ ($i \in \{1,2\}$) is empty or consists of formulas of the type $\circ A$, where A is an arbitrary formula; $\Box\Delta_i$ ($i \in \{1,2\}$) is empty or consists of formulas of the type $\Box A$, where A is an arbitrary formula. Formulas and sequents without temporal operators are called logical.

For the considered **PLTL** logic we consider the basic loop-type calculus $\mathbf{G_L PLTL}$ which is defined by the following postulates:

1. Logical axioms: $\Gamma, A \to \Delta, A$, where A is an arbitrary formula.
2. Logical rules: traditional invertible rules for logical connectives.
3. Temporal rules:

$$\frac{\Gamma \to \Delta}{\Pi, \circ\Gamma \to \Theta, \circ\Delta}\ (\circ), \qquad \frac{A, \circ\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to),$$

$$\frac{\Gamma \to \Delta, A; \Gamma \to \Delta, \circ\Box A}{\Gamma \to \Delta, \Box A}\ (\to \Box).$$

4. Loop-type axioms defined as follows:

   A quasi-primary sequent S' is a *looping sequent*, if (1) S' is not a logical axiom, (2) S' is above a sequent S on a branch of a derivation tree, (3) S is such that it subsumes S' ($S \succcurlyeq S'$ in notation), i.e., S' coincides with S or S' can be obtained from S by using the structural rule of weakening.

   A sequent S' is called a *degenerated sequent* (*d-sequent*, in short), if the one of the following two conditions is satisfied: (1) either S' is a looping sequent and there is no the right premiss of any application of $(\to \Box)$ between S and S', or S' consists of only propositional variables and is not a logical axiom; (2) S' is a looping sequent and there is the right premiss of an application of $(\to \Box)$ between S and S' but S is an ancestor of some d-sequent in the derivation.

   A looping sequent S' is called a *loop-type* (or *looping*) *axiom* if it is not a d-sequent. In this case the sequent S is called a *quasi-looping axiom*, if there is only one application of the rule $(\to \circ)$ between S and S'.

A sequent S is derivable in $\mathbf{G_L PLTL}$ (in notation $G_L PLTL \vdash S$) if we can construct the derivation V in $\mathbf{G_L PLTL}$ such that each branch of V ends either with a logical axiom or with a loop-type axiom. In the latter case we say that a "good loop" is obtained. The sequent S is not derivable in $\mathbf{G_L PLTL}$ (in notation $G_L PLTL \nvdash S$) if there exists a branch in V ending with a non-derivable logical sequent or containing a "bad looping" sequent, i.e., d-looping sequent which is not above the right premiss of the rule $(\to \Box)$. Looping sequents allow us to stop backward proof-search.

Example 1. Let $S = \circ\Box(\circ P \wedge \neg\Box A), \circ P \to \Box A$. Applying backward the temporal rules $(\to \Box)$, $(\circ)$, $(\Box \to)$ and logical rules $(\wedge \to)$, $(\neg \to)$ above the left premiss of application $(\to \Box)$ to S we get the sequent $S' = \circ\Box(\circ P \wedge \neg\Box A), \circ P, P \to \Box A$, i.e., $S \succeq S'$, but there is no right premiss of $(\to \Box)$ between S and S'. S' is a d-sequent. The conditions of looping axioms are not satisfied. Therefore, we have "bad looping", and $G_L PLTL \nvdash S$.

The loop-type calculus $\mathbf{G_L PLTL}$ corresponds to so called saturated calculi, considered in [4, 5, 6]. It follows from these works that $\mathbf{G_L PLTL}$ is sound and complete. It is obvious that all the rules of $\mathbf{G_L PLTL}$, except $(\circ)$, are invertible.

Lemma 1 [Reduction to primary (quasi-primary) sequents]. *By backward application of $(\Box \to)$, $(\to \Box)$, and logical rules (only logical rules), any sequent S can be reduced to a set of primary (quasi-primary) sequents $S_1, \ldots, S_n$ such that if $G_L PLTL \vdash S$, then $G_L PLTL \vdash S_i$, $1 \leqslant i \leqslant n$.*

Proof. The lemma is easily proved using the fact that rules used in the reduction are invertible and contraction-type rules are admissible in $\mathbf{G_L PLTL}$. □

2 Marked calculus

In this section, the marked calculus $\mathbf{G^m TL}$ for the considered $\mathbf{PLTL}$ is constructed. The calculus $\mathbf{G^m TL}$ allows us: (1) to construct loop-check free and backtracking free calculus; (2) to construct the terminating proof-search procedure for considered **PLTL**.

The calculus $\mathbf{G^m TL}$ contains the mark $*$ and marked propositional symbols P^* (along with non-marked ones). The marked formulas are defined as follows:

1. $(P^\alpha)^* = P^*$, where $\alpha \in \{\emptyset, *\}$;
2. $(A \odot B)^* = A^* \odot B^*$, where $\odot \in \{\supset, \vee, \wedge\}$;
3. $(\sigma A)^* = \sigma A^*$, where $\sigma \in \{\neg, \circ, \Box\}$.

The object of consideration in $\mathbf{G^m TL}$ is marked sequents of the shape $\Gamma \overset{\delta}{\to} \Delta$ where $\delta \in \{+, d\}$ along with non-marked sequents.

A quasi-primary sequent S is m-saturated (m-sat, in short, and denoted by S^m), if (1) S^m contains only marked propositional symbols and (2) S^m is not logical axiom.

A m-saturated sequent S^m is a positive ($+m$-sat, in short), if $S^m = \Gamma \overset{+}{\to} \Delta$. A m-saturated sequent S^m is a degenerated (dm-sat, in short), if $S^m = \Gamma \overset{d}{\to} \Delta$.

The marked calculus $\mathbf{G^m TL}$ is obtained from the looping calculus $\mathbf{G_L PLTL}$ by the following transformations:

1. The logical axioms are replaced by the sequents of the shape $\Gamma, A^{\alpha} \stackrel{\delta}{\to} \Delta, A^{\beta}$, where $\alpha, \beta \in \{\emptyset, *\}$, $\delta \in \{\emptyset, +, d\}$, A is an arbitrary formula.
2. The non-logical axioms (i.e., the looping axioms) are replaced by $+m$-sat sequents, which are now the new type of non-logical axioms.
3. The temporal rules $(\Box \to)$, $(\to \Box)$ are replaced by the following marked rules:

$$\frac{A, \bigcirc\Box A^{*}, \Gamma \stackrel{\delta}{\to} \Delta}{\Box A, \Gamma \stackrel{\delta}{\to} \Delta}\ (\Box^{*} \to), \qquad \frac{\Gamma \stackrel{d}{\to} \Delta, A; \quad \Gamma \stackrel{+}{\to} \Delta, \bigcirc\Box A^{*}}{\Gamma \stackrel{\delta}{\to} \Delta, \Box A}\ (\to \Box^{*}),$$

where $\delta \in \{\emptyset, +, d\}$; the conclusion of these rules is not m-sat sequent. The formula $\bigcirc\Box A^{*}$ is called the main side formula of the rules $(\Box^{*} \to)$ and $(\to \Box^{*})$.

A sequent S is derivable in $\mathbf{G^{m}TL}$ ($G^{m}\,TL \vdash S$ in notation), iff there exists the proof-search tree V such that each leaf of V there is either a logical axiom or a non-logical one, i.e., $+m$-sat sequent. In the opposite case, there exists a leaf containing either a logical sequent which is not derivable in propositional logic, or a dm-sat sequent. In this case the sequent S is non-derivable in $\mathbf{G^{m}TL}$ ($G^{m}\,TL \nvdash S$ in notation).

Example 2. Let $S = \bigcirc\Box(\bigcirc P \wedge \neg\Box A), \bigcirc P \to \Box A$, and let us consider the following backward proof-search in $\mathbf{G^{m}TL}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S_4 = P^{*}, \Box(\bigcirc P^{*} \wedge \neg\Box A^{*}) \stackrel{d}{\to}}{P, \bigcirc P^{*}, \bigcirc\Box(\bigcirc P^{*} \wedge \neg\Box A^{*}) \stackrel{d}{\to} A^{*}}\,(\bigcirc) \qquad S_3}{P, \bigcirc P^{*}, \bigcirc\Box(\bigcirc P^{*} \wedge \neg\Box A^{*}) \stackrel{d}{\to} \Box A^{*}}\,(\to \Box^{*})}{P, \Box(\bigcirc P^{*} \wedge \neg\Box A^{*}) \stackrel{d}{\to}}\,(\Box^{*} \to),\mathrm{rr}}{\bigcirc P, \bigcirc\Box(\bigcirc P^{*} \wedge \neg\Box A^{*}), P \stackrel{d}{\to} A}\,(\bigcirc) \qquad S_2}{\bigcirc P, \bigcirc\Box(\bigcirc P^{*} \wedge \neg\Box A^{*}), P \stackrel{d}{\to} \Box A}\,(\to \Box^{*})}{\Box(\bigcirc P \wedge \neg\Box A), P \stackrel{d}{\to}}\,(\Box^{*} \to),\mathrm{rr}}{\bigcirc\Box(\bigcirc P \wedge \neg\Box A), \bigcirc P \stackrel{d}{\to} A}\,(\bigcirc) \qquad S_1}{\bigcirc\Box(\bigcirc P \wedge \neg\Box A), \bigcirc P \to \Box A}\,(\to \Box^{*})$$

Here $rr = (\wedge \to), (\neg \to)$.

Since the sequent S_4 is a dm-sat sequent, we have $G^{m}\,TL \nvdash S$.

Lemma 2 [Looping property of $+m$-sat sequent]. *Let $G^{m}\,TL \vdash^{V} S$. Then each $+m$-sat sequent from a derivation tree V in $G^{m}\,TL$ satisfies all conditions of loop-type axiom.*

Proof. The proof follows from construction of V and the shape of temporal rules. □

3 Foundation of the calculus $\mathbf{G^{m}TL}$

Let $\widetilde{\mathbf{G}}^{\mathbf{m}}\mathbf{TL}$ be the calculus obtained from $\mathbf{G^{m}TL}$ by adding the loop-type axioms and non-marked modal rules $(\Box \to)$, $(\to \Box)$.

Lemma 3. *If* $\widetilde{G}^m TL \vdash^V S$*, then* $G^m TL \vdash^{V^*} S$.

Proof. The proof is carried out using induction on $n(V)$ different loop-type axioms in V. The case when $n(V) = 0$ is obvious. Let $n(V) > 0$. Consider any loop-type axiom S' in V. By definition of loop-type axioms, there exits a sequent S'' and a part of the given derivation V, which we denote by reduction $R(S'' \Rightarrow S')$, where S' can be obtained from S'' by the structural rule of weakening. Replacing the applications of non-marked rules $(\Box \rightarrow)$, $(\rightarrow \Box)$ in reduction $R(S'' \Rightarrow S')$ by applications of corresponding rules $(\Box^* \rightarrow)$, $(\rightarrow \Box^*)$, we get reduction of the sequent S'' to the $+m$-sat sequent S'^*. Hence, according to the inductive hypothesis, we have $G^m TL \vdash^{V^*} S$. $\Box$

Lemma 4. *If* $G^m TL \vdash^V S$*, then* $G_L PLTL \vdash^{V^*} S$.

Proof. The proof is carried out using induction on $p(V)$ different $+m$-sat sequents in V. The case when $p(V) = 0$ is obvious. Let $p(V) > 0$ and consider any $+m$-sat sequent S' in V. By the looping property in V, below S' there exits a sequent S'' such that S' can be obtained from S'' using the structural rule of weakening, moreover, there is the right premiss of $(\rightarrow \Box^*)$ between S' and S''. Therefore, S' can be regarded as looping axiom. Replacing the marked rules $(\Box^* \rightarrow)$, $(\rightarrow \Box^*)$ by non-marked ones and using the inductive hypothesis, we get $G_L PLTL \vdash^{V^*} S$. $\Box$

From Lemmas 3, 4, and soundness and completeness of $\mathbf{G_L PLTL}$, we get

Theorem 1. *The marked calculus* $\boldsymbol{G^m TL}$ *is sound and complete.*

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REZIUMĖ

Žymių metodas tiesinei laiko logikai
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Straipsnyje pateiktas korektiškas ir pilnas sekvencinis skaičiavimas tiesinei laiko logikai. Įrodymų baigtinumo nustatymui pateiktas naujas metodas naudojantis žymes. Šis metodas leidžia efektyviai taikyti modalines taisykles nagrinėjamai laiko logikai.

Raktiniai žodžiai: tiesinė laiko logika, sekvencinis skaičiavimas, išvedimo baigtinumas, žymės.