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A discrete limit theorem for the periodic Hurwitz zeta-function

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Abstract. In the paper, we prove a limit theorem of discrete type on the weak convergence of probability measures on the complex plane for the periodic Hurwitz zeta-function.

Keywords: Hurwitz zeta-function, limit theorem, probability measure, weak convergence.

Let $s = \sigma + it$ be a complex number, α, $0 < \alpha \leqslant 1$, be a fixed parameter, and let $\mathfrak{a} = \{a_m \colon m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and, by using the equality,

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{\alpha+l}{k}\right),$$

where $\zeta(s, \alpha)$ is the classical Hurwitz zeta-function, can be meromorphically continued to the whole complex plane with unique simple pole at the point $s = 1$ with residue

$$a \stackrel{\text{def}}{=} \frac{1}{k} \sum_{l=0}^{k-1} a_l.$$

If $a = 0$, then the function $\zeta(s, \alpha; \mathfrak{a})$ is entire one.

In [2, 4, 6] and [7], limit theorem on the weak convergence of probability measures on the complex plane $\mathbb{C}$ for the function $\zeta(s, \alpha; \mathfrak{a})$ with parameter α of various arithmetical types were obtained. In these works, the weak convergence for

$$\frac{1}{T} \mathrm{meas}\bigl\{t \in [0, T] \colon \zeta(\sigma + it, \alpha; \mathfrak{a}) \in A\bigr\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

where $\mathcal{B}(S)$ denotes the Borel σ-field of the space S, and $\mathrm{meas} A$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, was considered.

The theorems obtained are of continuous type because the imaginary part t can take arbitrary real values. The aim of this note is to prove a discrete limit theorem for the function $\zeta(s,\alpha;\mathfrak{a})$ when t takes values from the set $\{h_m\colon m\in\mathbb{N}_0\}$, where $h>0$ is a fixed number. Define the set

$$L(\alpha,h,\pi)=\biggl\{\bigl(\log(m+\alpha)\colon m\in\mathbb{N}_0\bigr),\frac{\pi}{h}\biggr\},$$

and the torus

$$\Omega=\prod_{m\in\mathbb{N}_0}\gamma_m,$$

where γ_m is the unit circle $\{s\in\mathbb{C}\colon |s|=1\}$ for all $m\in\mathbb{N}_0$. The torus Ω is a compact topological group, therefore, on $(\Omega,\mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$.

Denote by $\omega(m)$ the projection of an element $\omega\in\Omega$ to the coordinate space γ_m, $m\in\mathbb{N}_0$, and, on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, define the complex-valued random element $\zeta(\sigma,\alpha,\omega;\mathfrak{a})$ by the formula

$$\zeta(\sigma,\alpha,\omega;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m\omega(m)}{(m+\alpha)^{\sigma}},\quad \sigma>\frac{1}{2},$$

and denote by P_ζ the distribution of $\zeta(\sigma,\alpha,\omega;\mathfrak{a})$, i.e.,

$$P_\zeta(A)=m_H\bigl(\omega\in\Omega\colon \zeta(\sigma,\alpha,\omega;\mathfrak{a})\in A\bigr),\quad A\in\mathcal{B}(\mathbb{C}).$$

Theorem 1. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and that $\sigma>\frac{1}{2}$. Then the probability measure*

$$P_N(A)\overset{\text{def}}{=}\frac{1}{N+1}\#\bigl\{0\leqslant m\leqslant N\colon \zeta(\sigma+imh,\alpha;\mathfrak{a})\in A\bigr\},\quad A\in\mathcal{B}(\mathbb{C}),$$

converges weakly to the measure P_ζ as $N\to\infty$.

For the proof of Theorem 1, the following two lemmas involving the set $L(\alpha,h,\pi)$ are applied. Let

$$Q_N(A)\overset{\text{def}}{=}\frac{1}{N+1}\#\bigl\{0\leqslant m\leqslant N\colon \bigl((m+\alpha)^{-imh}\colon m\in\mathbb{N}_0\bigr)\in A\bigr\},\quad A\in\mathcal{B}(\Omega).$$

Lemma 1. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to the Haar measure m_H as $N\to\infty$.*

Proof of the lemma is given in [3, Lemma 2.3].

For $\mathfrak{a}_h=((m+\alpha)^{-ih}\colon|m\in\mathbb{N}_0)$, on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, define the transformation ψ_h by $\psi_h(\omega)=\mathfrak{a}_h\omega$, $\omega\in\Omega$. Then ψ_h is a measurable measure preserving transformation.

Lemma 2. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Then the transformation ψ_h is ergodic, i.e., if $A\in\mathcal{B}(\Omega)$ and $A_h=\psi_h(A)$ differ one from other at most by m_H-measure zero, then $m_H(A)=0$ or $m_H(A)=1$.*

Proof of the lemma is given in [3, Lemma 2.8].

The further proof of Theorem 1 can be divided into following parts: limit theorems for absolutely convergent Dirichlet series, approximation of the function $\zeta(\sigma, \alpha; \mathfrak{a})$ in the mean by absolutely convergent Dirichlet series, limit theorems for $\zeta(\sigma, \alpha; \mathfrak{a})$ and $\zeta(\sigma, \alpha, \omega; \mathfrak{a})$, and identification of the limit measure.

For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, define $\nu_n(m, \alpha) = \exp\bigl\{ -(\frac{m+\alpha}{n+\alpha})^{\sigma_1}\bigr\}$, where $\sigma_1 > \frac{1}{2}$ is a fixed number, and set

$$\zeta_n(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \nu_n(m, \alpha)}{(m+\alpha)^s}$$

and

$$\zeta_n(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) \nu_n(m, \alpha)}{(m+\alpha)^s}.$$

Then the latter series both are absolutely convergent for $\sigma > \frac{1}{2}$. Moreover, for $A \in \mathcal{B}(\mathbb{C})$, let

$$P_{N,h}(A) \stackrel{\text{def}}{=} \frac{1}{N+1} \#\bigl\{0 \leqslant m \leqslant N\colon \zeta_n(\sigma + imh, \alpha; \mathfrak{a}) \in A\bigr\},$$

and

$$P_{N,h,\omega}(A) \stackrel{\text{def}}{=} \frac{1}{N+1} \#\bigl\{0 \leqslant m \leqslant N\colon \zeta_n(\sigma + imh, \alpha, \omega; \mathfrak{a}) \in A\bigr\}.$$

Lemma 3. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ and that $\sigma > \frac{1}{2}$. Then $P_{N,h}$ and $P_{N,h,\omega}$ both converge weakly to the same probability measure P_n on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $N \to \infty$.*

Proof. The lemma is a result of the application of Lemma 1, Theorem 5.1 of [1], and the invariance of the Haar measure.

Lemma 2 is applied to show that, for almost all $\omega \in \Omega$, the estimate

$$\int_0^T \bigl|\zeta(\sigma + it, \omega; \mathfrak{a})\bigr|^2 = O(T), \quad T \to \infty,$$

is valid for $\sigma > \frac{1}{2}$. From this, using the Gallagher lemma, Lemma 1.4 of [5], we deduce that, for almost all $\omega \in \Omega$, the estimate

$$\frac{1}{N+1} \sum_{m=0}^{N} \bigl|\zeta(\sigma + imh, \alpha, \omega; \mathfrak{a})\bigr|^2 = O(1), \quad T \to \infty,$$

is valid for $\sigma > \frac{1}{2}$. Analogically, we obtain, for $\sigma > \frac{1}{2}$, the bound

$$\frac{1}{N+1} \sum_{m=0}^{N} \bigl|\zeta(\sigma + imh, \alpha; \mathfrak{a})\bigr|^2 = O(1), \quad T \to \infty.$$

Using the latter estimates and contour integration, we arrive to the following assertion.

Lemma 4. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ and that $\sigma > \frac{1}{2}$. Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left|\zeta(\sigma + imh, \alpha; \mathfrak{a}) - \zeta_n(\sigma + imh, \alpha; \mathfrak{a})\right| = 0,$$

and, for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \left|\zeta(\sigma + imh, \alpha, \omega; \mathfrak{a}) - \zeta_n(\sigma + imh, \alpha, \omega; \mathfrak{a})\right| = 0.$$

Let, for $A \in \mathcal{B}(\mathbb{C})$,

$$P_{N,\omega}(A) = \frac{1}{N+1} \#\left\{0 \leqslant m \leqslant N\colon \zeta(\sigma + imh, \alpha, \omega; \mathfrak{a}) \in A\right\}.$$

Lemma 5. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ and that $\sigma > \frac{1}{2}$. Then P_N and $P_{N,\omega}$ both converge weakly to the same probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $N \to \infty$.*

Proof. First we show that the family of probability measures $\{P_n\colon n \in \mathbb{N}\}$ tight. Therefore, by the Prokhorov theorem [1], this family is relatively compact. Hence, there exists a sequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $k \to \infty$. This, Lemmas 3 and 4, and Theorem 4.2 of [1] prove the lemma.

Proof of Theorem 1. In virtue of Lemma 5, it suffices to show that the measure P in Lemma 5 coincides with P_ζ.

Let A be an arbitrary continuity set of the measure P, i.e., $P(\partial A) = 0$, where ∂A is the boundary of A. Then Lemma 5 and the equivalent of weak convergence of probability measures in terms of continuity sets, Theorem 2.1 of [1], imply that

$$\lim_{n\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant m \leqslant N\colon \zeta(\sigma + imh, \alpha, \omega; \mathfrak{a}) \in A\right\} = P(A). \tag{1}$$

Now, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable θ by the formula

$$\theta(\omega) = \begin{cases} 1 & \text{if } \zeta(s, \alpha, \omega; \mathfrak{a}) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, the expectation $E\theta$ of the random variable θ is given by

$$E\theta = \int_\Omega \theta d\, m_H = m_H\left(\omega \in \Omega\colon \zeta(\sigma, \alpha, \omega; \mathfrak{a}) \in A\right) = P_\zeta(A). \tag{2}$$

Now we apply Lemma 1, and obtain by the classical Birkhoff–Khintchine ergodicity theorem that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \theta\left(\psi_h^m(\omega)\right) = E\theta \tag{3}$$

for almost all $\omega \in \Omega$. On the other hand, the definitions of the random variable θ and the transformation ψ_h yield the equality

$$\frac{1}{N+1}\sum_{m=0}^{N}\theta\big(\psi_h^m(\omega)\big) = \frac{1}{N+1}\#\big\{0 \leqslant m \leqslant N\colon \zeta(\sigma+imh,\alpha,\omega;\mathfrak{a}) \in A\big\}.$$

This together with (2) and (3) shows that, for almost all $\omega \in \Omega$,

$$\lim_{N\to\infty}\frac{1}{N+1}\#\big\{0 \leqslant m \leqslant N\colon \zeta(\sigma+imh,\alpha,\omega;\mathfrak{a}) \in A\big\} = P_\zeta(A).$$

Hence, in view of (1), we obtain that $P(A) = P_\zeta(A)$. Since the set A was arbitrary, we have that $P(A) = P_\zeta(A)$ for all continuity sets of the measure P. However, the continuity sets constitute the determining class, therefore, $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}(\mathbb{C})$.

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REZIUMĖ

Diskrečioji ribinė teorema periodinei Hurvico dzeta funkcijai

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Straipsnyje įrodyta diskretaus tipo ribinė teorema, silpnojo tikimybinių matų konvergavimo prasme, periodinei Hurvico dzeta funkcijai kompleksinėje plokštumoje.

Raktiniai žodžiai: Hurvico dzeta funkcija, ribinė teorema, silpnasis konvergavimas, tikimybinis matas.