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The basic solution ellipsoid method approach for the efficiency measurement problems

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Abstract. The efficiency is a measure of a performance of a decision making units (DMUs can be a firm, a person, an organization). The data envelopment analysis (DEA) is a data-driven non-parametric approach for measuring the efficiency of a set of DMUs. The DEA is a linear programming (LP) based technique which deals with the basic models (CCR, BCC, SBM, additive) of the efficiency evaluation. This paper presents basic solution ellipsoid method approach associated with some problems of initial basic solution and the steps of it.

Keywords: efficiency, data envelopment analysis, linear programming, basic ellipsoid.

Introduction

The productivity and efficiency are the two important concepts that are frequently utilised to measure performance of a decision making unit (DMU which can be a firm, a person, or an organization). Unfortunately, these two similar but different concepts have been used interchangeably. They are equivalent only when the technology exhibits constant returns to scale. The productivity of a unit is the ratio of the weighted sum of its outputs to the weighted sum of its inputs, and the efficiency is the ratio of the productivity of a DMU to the best productivity achieved.

The data envelopment analysis (DEA) is one of the most important approaches to measuring efficiency. It is a non-parametric method, pioneered and developed by authors [1, 2] to frontier analysis for measuring efficiency of a set of DMUs. Mathematically, the DEA utilizes a linear programming model which characterizes the relationship among multiple inputs and multiple outputs by an envelopment of the observed data to determine a piecewise linear empirical best practice frontier. Moreover, the non-parametric approach allows the identification of real “peers” constituting the basis of comparison for the DMU, thereby providing managerially valuable information for performance analysis and improvement. Conventionally, the DEA is used in ex post evaluation of actual performance, estimating an empirical best-practice frontier using minimal assumptions about the shape of the production space.

The linear programming (LP) is the fundamental problem of mathematical optimization.

The standard LP problem

$$\boldsymbol{C}^T \cdot \boldsymbol{X} = F \to \max, \qquad \boldsymbol{a}_j^T \cdot \boldsymbol{X} \leqslant b_j, \quad j \in \mathbb{M},\ \mathbb{M} = \{1, 2, \ldots, m\}, \tag{1}$$

where $\boldsymbol{C}^T = (c_1, c_2, \ldots, c_n)$, $\boldsymbol{X}^T = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{a}_j^T = (a_{j_1}, a_{j_2}, \ldots, a_{j_n})$. $\boldsymbol{C}^T\boldsymbol{X} = F$ is an objective function; $\boldsymbol{a}_j^T\boldsymbol{X} \leqslant b_j$ are linear constraints; $x_j \geqslant 0$, $i \in \mathbb{N}$, $\mathbb{N} = \{1, 2, \ldots, n\}$, $\mathbb{N} \subset \mathbb{M}$.

The three most significant classes of algorithms for solving LP problems are 3: pivot, ellipsoid and interior point methods [3]. The core of the well-known LP problems solution method is the Simplex method: an iterative procedure allowing to achieve the optimal solution of the problem in finite number of steps, by recalculating the coefficients of the (1) according to the certain rules. The practical efficiency of the Simplex method could not hide the frustrating fact that some variants of the Simplex method require an exponentially many steps to solve an LP problem. This is due to the fact that the Simplex method "travels" along the periphery of the domain area of feasible solutions, which must contain the optimal solution.

Among the other methods to solve the problem (1) there are the Karmarkar's (interior point) method [4] as well as the methods of circumscribed and inscribed ellipsoids [5]. The most common method which is competitive to or even more effective than the Simplex method is the polynomial Karmarkar's method.

This paper deals with the solution of the LP problem (1) by using the ellipsoid of the basic solution and the inverse matrix of the non-basic solution. The solution of the problem (1) does not involve any additional variables and no new conditions for variable non-negation are raised, which are often necessary in many of the algorithms for the solution of such task.

The method, first published in 1999 [6], unlike the methods of circumscribed and inscribed ellipsoids, uses the ellipsoid not for the determination of the area in which there is an optimal solution, but rather for the evaluation of the direction towards an optimal solution. The idea of the method described in this paper is realized by the following stages: a) an ellipsoid is constructed from the basic constraints of the solution; b) an optimization direction, which guarantees the increase of the objective function and remains within the area of the feasible solutions, is evaluated using the local ellipsoid; c) the vector of the objective function is oblique projected into the subset of the non-basic solution by using the inverse matrix of the non-basic solution.

1 An ellipsoid of the basic solution

Suppose that the basic feasible non-optimal solution is found. The system is not reduced assuming that it is composed from the first n ($n < m$) constraints. The basic solution is fully described by the vector matrix of the base that consists of constraints $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$, its inverse matrix $\boldsymbol{A}^{-1}$ and vectors $\boldsymbol{B} = (b_1, b_2, \ldots, b_n)$ and $\boldsymbol{C}$. It is known that the basic solution $\boldsymbol{X}_B = (\boldsymbol{A}^T)^{-1}\boldsymbol{B}$ (where $\boldsymbol{C}^T\boldsymbol{X}_B = F_B$) is optimal, if the condition

$$\boldsymbol{A}^{-1}\boldsymbol{C} = \boldsymbol{P} \geqslant 0 \tag{2}$$

is valid, i.e., if $p_i \geqslant 0$, $\forall i \in \mathbb{N}$, $\mathbb{N} \subset \mathbb{M}$, where $\boldsymbol{P}^T = (p_1, p_2, \ldots, p_n)$. The ellipsoid of the basic solution will be the function

$$Q(\boldsymbol{X}) = \sum_{j \in \mathbb{N}} \left(\boldsymbol{a}_j^T\boldsymbol{X} - b_j\right)^2, \tag{3}$$

which is the least square method with respect to all the planes (hyperplanes) of the basic solution constraints. It is not difficult to see that (3) is positively defined square shape with $\nabla Q(\boldsymbol{X}) = \boldsymbol{A}\boldsymbol{A}^T\boldsymbol{X} - \boldsymbol{A}\boldsymbol{B}$. The center of the ellipsoid $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$ is the solution of equation $\nabla Q(\boldsymbol{X}) = \boldsymbol{0}$, which is the same as the basic solution. Suppose that $F_1 = \boldsymbol{C}^T\boldsymbol{X} > F_B$, then the solution of the problem

$$Q(\boldsymbol{X}) + \lambda\big(\boldsymbol{C}^T\boldsymbol{X} - F_1\big) \to \min$$

where λ is the Lagrange multiplier, $\lambda < 0$. Geometrical meaning: the point O, at which the plane F_1 touches non-zero volume ellipsoid $Q(\boldsymbol{X})$, is:

$$\boldsymbol{X}_O = \boldsymbol{X}_B - \lambda\big(\boldsymbol{A}^T\big)^{-1}\boldsymbol{A}^{-1}\boldsymbol{C} = \boldsymbol{X}_B + \beta\big(\boldsymbol{A}^T\big)^{-1}\boldsymbol{P}, \quad \beta = -\lambda > 0.$$

On the other hand, X_O is the solution of the equation $\nabla Q(\boldsymbol{X}) = \beta\boldsymbol{C}$, i.e., within the radius

$$\boldsymbol{X}_O = \boldsymbol{X}_B + \beta\big(\boldsymbol{A}^T\big)^{-1}\boldsymbol{P}, \quad \beta > 0.$$

The vector $\boldsymbol{P}_L = (\boldsymbol{A}^T)^{-1}\boldsymbol{P}$ will be called the touching-vector of the optimal solution. The gradient of the ellipsoid is collinear to the vector $\boldsymbol{C}$ of the target function (1). The basic solution is optimal, if the condition $\boldsymbol{A}^T\boldsymbol{X} - \boldsymbol{B} \geqslant 0$ is satisfied at the point X_O. Really

$$\big(\boldsymbol{A}^T\boldsymbol{X} - \boldsymbol{B}\big)_O = \boldsymbol{A}^T\big(\boldsymbol{X}_B + \beta(\boldsymbol{A}^T)^{-1}\boldsymbol{P}\big) - \boldsymbol{B} = \boldsymbol{A}^T\boldsymbol{X}_B + \beta\boldsymbol{A}^T\big(\boldsymbol{A}^T\big)^{-1}\boldsymbol{P} - \boldsymbol{B} = \beta\boldsymbol{P} \geqslant 0.$$

That is that the condition (2) is satisfied. The condition $\boldsymbol{A}^T\boldsymbol{X} - \boldsymbol{B} \geqslant 0$ means that at any point of the radius

$$\boldsymbol{X}_O = \boldsymbol{X}_B + \beta\big(\boldsymbol{A}^T\big)^{-1}\boldsymbol{P}, \quad \beta > 0$$

none of the constraints are strictly satisfied.

The point O will be called the image of the optimal solution within the ellipsoid. Which point would be the image of the non-optimal solution within the ellipsoid? Suppose that within the k-axis ($k \in \mathbb{N}$) the constraint of the basic solution is satisfied strictly, i.e., $\boldsymbol{a}_k^T\boldsymbol{X} - b_k < 0$. The condition $\boldsymbol{A}^T\boldsymbol{X} - \boldsymbol{B} \geqslant 0$ is not satisfied, therefore the basic solution is not optimal. We will use a function

$$Q(\boldsymbol{X})_{-k} = \sum_{j \in \mathbb{N}, j \neq k} \big(\boldsymbol{a}_j^T\boldsymbol{X} - b_j\big)^2$$

composed only from strictly non-satisfied constraints.

$$\nabla Q(\boldsymbol{X})_{-k} = \boldsymbol{A}_{-k}\big(\boldsymbol{A}^T\big)_{-k}\boldsymbol{X} - \boldsymbol{A}_{-k}\boldsymbol{B}_{-k}$$

where $\boldsymbol{A}_{-k} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k = 0, \ldots, \boldsymbol{a}_n)$ is a matrix $\boldsymbol{A}$, in which the vector $\boldsymbol{a}_k$ is exchanged with the zero vector. $\boldsymbol{B}_{-k} = (b_1, b_2, \ldots, b_k = 0, \ldots, b_n)$ is the vector $\boldsymbol{B}$ with the kth zero component. Then $(\nabla Q(\boldsymbol{X})_{-k})_O = \beta\boldsymbol{A}_{-k}\boldsymbol{P}_{-k}$, where $(\boldsymbol{P}_{-k})^T = (p_1, p_2, \ldots, p_k = 0, \ldots, p_n)$ is a vector, the kth element is a zero element. The vector $\beta\boldsymbol{A}_{-k}\boldsymbol{P}_{-k}$ is a vector of a such hyperplane D that gives a non-optimal solution. $\boldsymbol{X}_N$ is the vector of the coordinates of the point N at which the ellipsoid $Q(\boldsymbol{X})$ touches the hyperplane D: $\boldsymbol{X}_N = \boldsymbol{X}_B + \beta(\boldsymbol{A}^T)^{-1}\boldsymbol{P}_{-k}$. The point N will be called the image of the non-optimal solution within the ellipsoid, and a vector $\boldsymbol{D}_L = \beta(\boldsymbol{A}^T)_{-k}\boldsymbol{P}_{-k}$ is the touching-vector of the non-optimal solution.

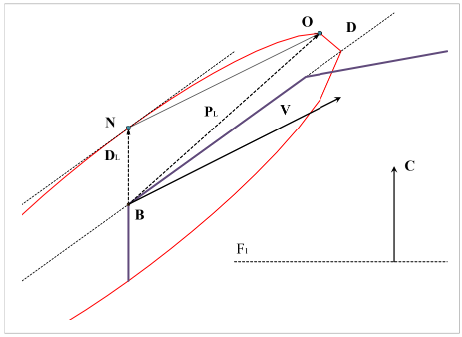

Fig. 1. The next step of the direction vector $\boldsymbol{V}$.

After the values of the function $Q(\boldsymbol{X})$ are determined at the points O and N, it is possible to make the following conclusion: the point N (the image of the non-optimal solution within the ellipsoid) belongs to all the planes of the strictly satisfied constraints. It is obvious that values of the function $Q(\boldsymbol{X})$ at the points O and N satisfy inequality $(Q(\boldsymbol{X}))_O > (Q(\boldsymbol{X}))_N$, i.e., the point N belongs to the ellipsoid of the smaller volume. By choosing the multiplier $\alpha > 0$ for the vector $\beta(\boldsymbol{A}^T)^{-1}\boldsymbol{P}_{-k}$, it is possible to achieve that $(Q(\boldsymbol{X}))_O = (Q(\boldsymbol{X}))_N$. Then $\alpha^2 = \frac{\eta_O}{\eta_N} > 1$, where $\eta_N \neq 0$.

We will construct another vector of a new optimization direction: $\boldsymbol{V} = \boldsymbol{P}_L - \alpha\boldsymbol{D}_L$, i.e., a vector from non-optimal image to optimal image of solution within the ellipsoid (see Fig. 1). It is possible to prove that the vector $\boldsymbol{V}$, where $\alpha^2 \in \left(1, \frac{\eta_O}{\eta_N}\right)$, satisfies the following conditions:

$$\boldsymbol{C}^T\boldsymbol{V} > 0, \qquad \boldsymbol{a}_j^T\boldsymbol{V} < 0, \quad j \in \mathbb{N} \tag{4}$$

i.e., guarantees the increase of the target function (1), and the point S ($\boldsymbol{X}_S = \boldsymbol{X}_B + \varepsilon\boldsymbol{V}, \varepsilon > 0$) belongs to the region of the feasible solutions.

2 Method of solution

Let us allow that $\boldsymbol{X}_B$ is feasible non-optimal basic solution of the task (1). Before creating the basis of the new solution, i.e., we start with the fact that the matrices of the new base are the unitary matrices, $\boldsymbol{A} = \boldsymbol{A}^{-1} = \boldsymbol{I}_n$. We will use a vector $\boldsymbol{U}$, showing the occupancy of the matrices of the new base $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ with the constraint vectors. If $u_k = 0$, then the kth columns of $\boldsymbol{A}$ ir $\boldsymbol{A}^{-1}$ are free, i.e., $\boldsymbol{a}_k = (0, 0, \dots, a_k = 1, \dots, 0)$. If $u_k = r$, where $r \in \mathbb{M}$, the kth column of $\boldsymbol{A}$ is occupied and $\boldsymbol{a}_k = (a_{r_1}, a_{r_2}, \dots, a_{r_n})$. Before making the basis of the new solution, the vector $\boldsymbol{U}$ is a zero vector: $\boldsymbol{U} = \boldsymbol{0}$. Assume that $\boldsymbol{X}^{(z)}$ is a solution after z steps of the making of the new basis $\boldsymbol{X}^{(0)} = \boldsymbol{X}_B$.

$z = 1$. In the radius $\boldsymbol{X} = \boldsymbol{X}^{(0)} + \boldsymbol{V}^{(0)}t$, $t > 0$ the closest constraint is searched. The possible cases of this process are not discussed here. Let us assume, that nearest constraint a_m is appended to the base, to the kth column of the matrix $\boldsymbol{A}$. Then $u_k = m$. The vectors, except for the kth row, of the inverse matrix $\boldsymbol{A}^{-1}$are orthogonal to the kth column of the matrix $\boldsymbol{A}$. Therefore, the row-vectors are going to be on the plane of the constraint a_m. The set of linear vectors appended to the vector $\boldsymbol{C}$ of the objective function is going to make the $n-1$-dimensional vector, that belongs to the vector space $\mathbb{R}^n$.

$$\boldsymbol{A}^{-1}\boldsymbol{C}^T = \boldsymbol{C}^* \quad \Rightarrow \quad \boldsymbol{C}^*\boldsymbol{I}_{n-z} = \boldsymbol{C}^*_{n-z} \quad \Rightarrow \quad \left(\boldsymbol{C}^*_{n-z}\right)^T \boldsymbol{A}^{-1} = \boldsymbol{C}^{(z)}_{n-z},$$

where $\boldsymbol{I}_{n-z}$ is the identity matrix, that has z number of null rows; $\boldsymbol{C}^{(z)}_{n-z}$ is the non-orthogonal (oblique) projection of the vector $\boldsymbol{C}$ in the $n-z$ vector space after the z steps. The oblique projection enables to get the optimal solution using the surface of the polyhedron, and not the edges of it, like in the Simplex method.

$z = \{2, 3, \ldots, n\}$. In order to maintain the solution within the region of the feasible solutions, the new search direction of the optimal solution is evaluated by projecting the vector $\boldsymbol{C}$ of the objective function towards $k = (n-z+1)$-dimensional edge, composed from the z constraint vectors included into the new base.

It is possible to prove that the optimization direction vector $\boldsymbol{V}^{(z)} = \boldsymbol{C}^T(\boldsymbol{O}_k, (\boldsymbol{A}^{-1})^{(z)})$ obtained after z steps satisfies conditions (4).

Algorithm 1 Short algorithm for the solution of the task using the basic ellipsoid method is presented below:

Begin

1: if $\boldsymbol{V} = \boldsymbol{0}$, then goto **2**;

within the radius $\boldsymbol{X} = \boldsymbol{X}^{(z)} + \boldsymbol{V}^{(z)}t$, $t > 0$ the closest constraint is searched;

vector $\boldsymbol{V}$ evaluated:

- using basic solution ellipsoid, if $z = 1$;
- using projective matrix of the non-basic solution $\boldsymbol{A}^{-1}$, if $1 < z \leqslant n$;
- goto **1**.

2: End.

The efficiency evaluation tasks, written by the DEA models, are characterized by the features suitable for the Basic Solution Ellipsoid (BSE) usage: 1) the number of items being assessed is quite large (up to the tens and hundreds place), and individual tasks shall be formulated and solved for each item; 2) in order to comprehensively assess the efficiency of each DMU, several models have to be employed: radial models of resources and product technical efficiency, directional efficiency, distributional and other models; 3) not all models have a starting zero-solution allowed, therefore, in such cases, a dual model is used. The circumstances, indicated above, are quite appropriate for the use of the BSE methods.

Algorithm 2 The steps of the LP and BSE tasks.

The LP task (1) is being solved. The LP task starts always from the zero-solution.

Begin

1: The algorithm of the BSE method is being applied until the following is reached:

- feasible optimal solution. Goto **2**.
- non-feasible non-optimal solution. Goto **1**.
- non-feasible optimal solution. The feasible optimal solution is within the vectors cone of reached solution. Therefore, one has to return to the feasible non-optimal, which is close to the feasible optimal solution. Goto **1**.

2: End

In order to compare the solving times of the BSE and the Simplex methods, the efficiencies of basketball players playing in the championship of Lithuanian Basketball league were calculated ($m = 96$, $n = 15$). The advantage of the BSE method is determined by the conversion speed of an incomplete inverse matrix.

3 Conclusions

In this paper the approach of the basic solution of the ellipsoid method in order to solve the efficiency measurement problems, i.e., the LP problems, is proposed. The method is based on the approximation of the basic solution using the non-linear ellipsoid method. The direction of the search of the optimal solution is determined using the images of the optimal and the current non-optimal solutions in the ellipsoid. Using the oblique projection, the search path going on the surface of the polyhedron, that is formed by the constraints of the system, is found. The calculations confirm the assumptions of the basic solution of the ellipsoid method and its use.

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REZIUMĖ

Bazinio sprendinio elipsoido metodo taikymas efektyvumo vertinimo uždaviniuose

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Efektyvumas yra panašios veiklos dalyvių (žmonių, įmonių, organizacijų) produktyvumo įvertinimo matas. Sprendimo DEA yra duomenimis neparametrinis efektyvumo vertinimo metodas. DEA vertinimo modeliai (*angl.* CCR, BCC, SBM, adityvusis) – tiesinio programavimo uždaviniai. Darbe pateikiamas bazinio sprendinio elipsoido metodas pasirodė tinkamas pradinio sprendinio ir sprendimo laiko požiūriu.

Raktiniai žodžiai: efektyvumas, duomenų apgaubimo analizė, tiesinis programavimas, bazinis elipsoidas.