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Fast Fourier transform revisited

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Abstract. Using FFT (fast Fourier transform), it is assumed, that some signal samples in a respective period N are updated by a sensor in real time. It is urgent for every new signal sample to have new frequency samples (f.s.). The idea is that FFT should not be recalculated with every new signal sample, it is needed just to modify it, when the new sample replaces the old one.

Keywords: digital signal processing (DSP), discrete Fourier transform (DFT).

1 Introduction

It is known [1, 2] that some problems, encountered with FFT applications to measured samples of signals, are not generally understood, e.g. if some samples or even one sample in the given period is replaced by new samples or one sample, respectively, and for each such case we have to obtain a new spectrum immediately. In such a case, it is needed to modify DFT in order to recalculate on-line only some products of the Fourier 'code matrix'.

2 Statement of the problem

We consider a discrete-time finite duration real-valued signal $\{x(n)\}$ of length L (i.e., $\{x(n)\} = 0$ for $n < 0$ and $n \geqslant L$) that has the Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}, \quad \forall \omega \in \overline{0, 2\pi}. \tag{1}$$

Here j is the imaginary unit. When we sample $\{X(\omega)\}$ at equally spaced frequencies $\omega_k = 2\pi k/N$, $\forall k \in \overline{0, N-1}$, with $N \geqslant L$, the resultant samples are as follows [2]:

$$X(k) \equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad \forall k \in \overline{0, N-1}. \tag{2}$$

For convenience, the upper index in the sum has been increased from $L-1$ to $N-1$ since $\{x(n)\} = 0$ for $n \geqslant L$. Here N is the total number of samples of the basic real-valued signal $x(n)$, $\forall n \in \overline{0, N-1}$ under consideration. The relation in (2) is called DFT of $\{x(n)\}$ and is used for transforming the sequence $\{x(n)\}$ into a sequence of f.s. $\{X(k)\}$ of length N.

Assume that at any moment t_l the network of sensors is simultaneously evaluated the set of DFT samples $\{X(n)\}$ by processing the signal samples $\{x(n)\}$. At time moment t_{l+1} the new set of current samples $\{x_{new}(n)\}$ enter memory replacing the previous samples $\{x_{old}(n)\}$. For the moment it is determined that most of signal's samples are equivalent to the previous samples. Only about five percent or less of current samples $\{x_{new}(n)\}$ are different. In such a case, it is not efficient to recalculate the basic spectrum samples anew, especially, when the calculations speed is a main issue. Therefore, it is important to work out an approach for modifying DFT algorithm in order to decrease the calculation time significantly.

The aim of the paper is to work out an approach that would update the f.s. $X(k)$, $\forall k \in \overline{0, N-1}$ as fast as possible with a new sensor's samples that emerge and replace the previous ones.

3 On-line FFT

Let us formulate now the corollary for recalculating the f.s. $X(k)$, $\forall k \in \overline{0, N-1}$, when new sensor's samples appear in the given period N while the respective old samples of a signal $\{x(n)\}$ vanish.

Corollary 1. *The current f.s. $X(k)$, $\forall k \in \overline{0, N-1}$ are calculated by the recursive formula*

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \underbrace{X(N-1)}_{current\ f.s.} \end{bmatrix} = \begin{bmatrix} X(0) + \varDelta X(0) \\ X(1) + \varDelta X(1) \\ \vdots \\ \underbrace{X(N-1)}_{previous\ f.s.} + \underbrace{\varDelta X(N-1)}_{correction} \end{bmatrix}, \tag{3}$$

assuming that some new samples emerge and replace the previous ones. Here spectrum samples are calculated by the ordinary FFT:

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \underbrace{X(N/2-1)}_{previous\ f.s.} \end{bmatrix} = \begin{bmatrix} X_1(0) + W_N^{(0)} X_2(0) \\ X_1(1) + W_N^{(1)} X_2(1) \\ \vdots \\ X_1(N/2-1) + W_N^{(N/2-1)} X_2(N/2-1) \end{bmatrix} \tag{4}$$

if $0 \leqslant k \leqslant N/2-1$ *and*

$$\begin{bmatrix} X(N/2) \\ X(N/2+1) \\ \vdots \\ \underbrace{X(N-1)}_{previous\ f.s.} \end{bmatrix} = \begin{bmatrix} X_1(0) - W_N^{(0)} X_2(0) \\ X_1(1) - W_N^{(1)} X_2(1) \\ \vdots \\ X_1(N/2-1) - W_N^{(N/2-1)} X_2(N/2-1) \end{bmatrix} \tag{5}$$

if $N/2 \leqslant k \leqslant N-1$,

$$X_1(k) = \sum_{m=0}^{N/2-1} x(2m) W_{N/2}^{(mk)}, \qquad X_2(k) = \sum_{m=0}^{N/2-1} x(2m+1) W_{N/2}^{(mk)} \tag{6}$$

$\forall k \in \overline{0, N-1}$. *Here* $x(2m)$, $x(2m+1)$ *for varying* m *are samples of* $\{x(n)\}$, $W_N^{(nk)} = e^{-j2\pi nk/N}$.

Proof of Corollary 1. Current spectrum samples can be represented as follows:

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \underbrace{X(N/2-1)}_{current\ f.s.} \end{bmatrix} = \begin{bmatrix} \tilde{X}_1(0) + W_N^{(0)} \tilde{X}_2(0) \\ \tilde{X}_1(1) + W_N^{(1)} \tilde{X}_2(1) \\ \vdots \\ \tilde{X}_1(N/2-1) + W_N^{(N/2-1)} \tilde{X}_2(N/2-1) \end{bmatrix} \tag{7}$$

if $0 \leqslant k \leqslant N/2-1$ or

$$\begin{bmatrix} X(N/2) \\ X(N/2+1) \\ \vdots \\ \underbrace{X(N-1)}_{current f.s.} \end{bmatrix} = \begin{bmatrix} \tilde{X}_1(0) - W_N^{(0)} \tilde{X}_2(0) \\ \tilde{X}_1(1) - W_N^{(1)} \tilde{X}_2(1) \\ \vdots \\ \tilde{X}_1(N/2-1) - W_N^{(N/2-1)} \tilde{X}_2(N/2-1) \end{bmatrix} \tag{8}$$

if $N/2 \leqslant k \leqslant N-1$,

$$\tilde{X}_1(k) = \sum_{m=0}^{N/2-1} \tilde{x}(2m) W_{N/2}^{(mk)}, \qquad \tilde{X}_2(k) = \sum_{m=0}^{N/2-1} \tilde{x}(2m+1) W_{N/2}^{(mk)}, \tag{9}$$

$\forall k \in \overline{0, N-1}$. Here $\tilde{x}(2m)$, $\tilde{x}(2m+1)$ for varying m are new samples of $\{x(n)\}$. However,

$$\sum_{m=0}^{N/2-1} \tilde{x}(2m) = \sum_{m=0}^{N/2-1} \bigl\{x(2m) + \Delta x(2m)\bigr\} \tag{10}$$

and

$$\sum_{m=0}^{N/2-1} \tilde{x}(2m+1) = \sum_{m=0}^{N/2-1} \bigl\{x(2m+1) + \Delta x(2m+1)\bigr\}. \tag{11}$$

Here $\Delta x(2m)$, $\Delta x(2m+1)$ are corrections for old samples. Then, it follows

$$\begin{aligned} \tilde{X}_1(k) &= X_1(k) + \underbrace{\sum_{m=0}^{N/2-1} \Delta x(2m) W_{N/2}^{(mk)}}_{correction}, \\ \tilde{X}_2(k) &= X_2(k) + \underbrace{\sum_{m=0}^{N/2-1} \Delta x(2m+1) W_{N/2}^{(mk)}}_{correction}. \end{aligned} \tag{12}$$

The corollary is proven by substituting right-hand side terms of equations (3) into (7) and (8), respectively. $\square$

4 Example

The discrete-time periodic signal $x(n) = \{\dots 24, 8, 12, 16, 20, 6, 10, 14, \dots\}$. By inspection, the period $N = 8$. The DFT is computed by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ \underbrace{X(7)}_{current\ f.s.} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_8 & W_8^2 & \dots & W_8^7 \\ 1 & W_8^2 & W_8^4 & \dots & W_8^{14} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_8^7 & W_8^{14} & \dots & W_8^{49} \end{bmatrix} \begin{bmatrix} 24 \\ 8 \\ 12 \\ 16 \\ 20 \\ 6 \\ 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 110 \\ 4-4.83j \\ 22+16j \\ 4-0.83j \\ 22 \\ 4+0.83j \\ 22-16j \\ 4+4.83j \end{bmatrix}, \tag{13}$$

using the known Fourier 'code' matrix with twiddle factors as follows:

$$W_8 = W_8^9 = W_8^{25} = W_8^{49} = a(1-j), \qquad W_8^2 = W_8^{10} = W_8^{18} = W_8^{42} = -j,$$
$$W_8^3 = W_8^{35} = b(1+j), \qquad W_8^4 = W_8^{12} = W_8^{20} = W_8^{28} = W_8^{36} = -1,$$
$$W_8^5 = W_8^{21} = b(1-j), \qquad W_8^6 = W_8^{14} = W_8^{30} = j,$$
$$W_8^7 = W_8^{15} = a(1+j), \qquad W_8^8 = W_8^{16} = W_8^{24} = 1.$$

Here $a = 0.7071$ and $b = -a$. In this paper the spectrum samples are obtained also using FFT of the form

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ \underbrace{X(7)}_{current\ f.s.} \end{bmatrix} = \begin{bmatrix} G(0) + W_8^0 H(0) \\ G(1) + W_8^1 H(1) \\ G(2) + W_8^2 H(2) \\ G(3) + W_8^3 H(3) \\ G(0) - W_8^0 H(0) \\ G(1) - W_8^1 H(1) \\ G(2) - W_8^2 H(2) \\ \underbrace{G(3) - W_8^3 H(3)}_{current\ FFT\ terms} \end{bmatrix} = \begin{bmatrix} 110 \\ 4-4.83j \\ 22+16j \\ 4-0.83j \\ 22 \\ 4+0.83j \\ 22-16j \\ 4+4.83j \end{bmatrix}, \tag{14}$$

where

$$G(k) = \sum_{m=0}^{3} x(2m) W_4^{2mk}, \qquad H(k) = \sum_{m=0}^{3} x(2m+1) W_4^{(2m+1)k}, \quad \forall k \in (0,7). \tag{15}$$

The same results we obtain launching Matlab's function *fft*: $X = \mathit{fft}([24, 8, 12, 16, 20, 6, 10, 14], 8)$. Then, we changed the data set $x(n)$ as follows $x(n) = \dots 24, 8, 12, 16, 20, 10, 10, 14, \dots$. The correction vector in time domain is of the form $\Delta x(n) = \dots 0, 0, 0, 0, 0, 4, 0, 0, \dots$. We calculate correction terms by (13) and using Matlab function: $\mathit{fft}([00000400], 8)$. Corrections of frequency values are: $\Delta X(0) = 4$, $\Delta X(1) = -2.828 + 2.828j$, $\Delta X(2) = -4j$, $\Delta X(3) = 2.828 + 2.828j$, $\Delta X(4) = -4$, $\Delta X(5) = 2.828 -$

$2.828j$, $\mathit{\Delta} X(6) = 4j$, $\mathit{\Delta} X(7) = -2.828 - 2.828j$. Continuing, we can write for current f.s. the recursive relationship

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix}}_{current} = \begin{bmatrix} G(0) + W_8^0 H(0) + \mathit{\Delta} X(0) \\ G(1) + W_8^1 H(1) + \mathit{\Delta} X(1) \\ G(2) + W_8^2 H(2) + \mathit{\Delta} X(2) \\ G(3) + W_8^3 H(3) + \mathit{\Delta} X(3) \\ G(0) - W_8^0 H(0) + \mathit{\Delta} X(4) \\ G(1) - W_8^1 H(1) + \mathit{\Delta} X(5) \\ G(2) - W_8^2 H(2) + \mathit{\Delta} X(6) \\ \underbrace{G(3) - W_8^3 H(3)}_{previous\ FFT\ terms} \quad \underbrace{+\mathit{\Delta} X(7)}_{correction\ terms} \end{bmatrix} = \begin{bmatrix} 114 \\ 1.17 - 2j \\ 22 + 12j \\ 6.83 + 2j \\ 18 \\ 6.83 - 2j \\ 22 - 12j \\ 1.17 + 2j \end{bmatrix}, \tag{16}$$

where previous spectrum samples (14) and corrections, calculated above, were used. Then, we checked up with *fft*$([24, 8, 12, 16, 20, 10, 10, 14], 8)$. The results obtained by *fft* are coincident with (16). Now, it is important to determine the computational burden, needed to obtain current spectrum values by recursive algorithm (3). Only N multiplication operations are necessary to calculate the correction terms $\mathit{\Delta} X(k)$, $\forall k \in \overline{0, N-1}$, because all the cells of vector $\mathit{\Delta} x(n)$ are filled with zeros except one. Later we need N addition operations, used to add the respective complex-valued elements in the cells of previous f.s. vector to that in the correction term one. So, to calculate the spectrum anew by recursive algorithm (3) we need N CMADs (complex multiplications and additions) and N extra addition operations, in total, after a new sample emerges and the former one vanishes. On the other hand, the ordinary FFT requires $N \log_2 N$ CMADs. It follows from the 8-point DFT example with real-valued samples set, that 8-point FFT, requires 24 CMADs and 8 extra complex addition operations if one new sample comes in.

5 Conclusions

For discrete-time signals the DFT coefficient values have been proposed to recursively determine if one new signal sample or new small portion of samples emerge in the given period N of a realization $\{x(n)\}$ replacing the old one sample or old portion of samples, respectively. The number of operations for their speedy calculating is essentially reduced by the original recursive expression (3) in comparison with the ordinary FFT procedure used only in the case of fixed values of samples $x(n)$, $\forall\ n \in \overline{0, N-1}$ in a fixed period N. An example of 8-point DFT, presented here, has shown us the efficiency of the recursive approach, too. The recursive algorithm could be effective in real-time applications for very large N values ($N \geqslant 2^{10}$) because it renders us a possibility to calculate a varying spectrum on-line with greatly less number of CMADs as compared with well-known Cooley-Tukey FFT.

References

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REZIUMĖ

Peržiūrint diskrečiąją Furjė transformaciją
R. Pupeikis

Tariama, kad taikant diskrečiąją Furjė transformaciją, signalo atskaitų apdorojimui skaitmeniškai, kai kurios jo atskaitos esti jutiklių, veikiančių realiu laiku, pakeičiamos naujomis atskaitomis. Būtina kiekvienai naujai atsiunčiamai atskaitai skaičiuoti naują spektrą. Tokiu atveju siūloma neperskaičiuoti spektrą naujai, o jį modifikuoti rekurentiškai, žymiai sutaupant aritmetinių operacijų skaičių. Pateiktas 8-atskaitų spektro skaičiavimo įprastiniu ir rekurentiniu būdais diskrečiąja Furjė transformacija pavyzdys.

Raktiniai žodžiai: skaitmeninis signalų apdorojimas (SSA), diskrečioji Furjė transformacija (DFT).