Lietuvos matematikos rinkinys, Vol. 60, 2019 ISSN 0132-2818
Proc. of the Lithuanian Mathematical Society, Ser. A eISSN 2335-898X
https://doi.org/10.15388/LMR.A.2019.14951 pages 27–32

The minimizer for the second order differential problem with the integral condition

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Abstract. In this paper, we investigate the best fit solution for the second order differential problem with one initial and other integral conditions. We obtain the representation of that minimizer and present an example.

Keywords: integral condition, least squares solution, Moore–Penrose inverse, minimizer.

1 Introduction

In the paper [2], there was studied the *best fit solution* to the second order differential problem with one initial condition and other nonlocal two point condition.

Let us now continue the investigation taking the integral condition

$$-u'' = f(x), \quad x \in [0,1], \tag{1}$$

$$u(0) = g_1, \qquad u(1) = \gamma \int_0^\xi u(x)\,dx + g_2, \tag{2}$$

where $f \in L^2[0,1]$, $g_1, g_2, \gamma \in \mathbb{R}$ and $\xi \in (0,1)$. If the parameter $\gamma = 0$, this problem becomes *classical*, it is uniquely solvable and has the Green's function

$$G^{cl}(x,y) = \begin{cases} y(1-x), & y \leqslant x, \\ x(1-y), & y \geqslant x. \end{cases} \tag{3}$$

For nonvanishing values of the parameter γ, the problem (1)–(2) becomes *nonlocal*. If $\gamma\xi^2 \neq 2$ [3], it also has the unique solution, which is of the form

$$u = g_1 + \frac{(\gamma\xi - 1)g_1 + g_2 + \gamma \int_0^1 \int_0^\xi G^{cl}(t,y) f(y)\,dt\,dy}{1 - \gamma\xi^2/2} x + \int_0^1 G^{cl}(x,y) f(y)\,dy.$$

For $\gamma\xi^2 = 2$, the problem (1)–(2) does not have the unique solution. This is the case that we are going to study below and obtain the so called *best fit solution*.

2 The vectorial problem

As in [2], we also consider the equivalent vectorial form

$$\boldsymbol{L}u = \boldsymbol{f} \tag{4}$$

of the problem (1)–(2), take the inner product $(\boldsymbol{f}, \tilde{\boldsymbol{f}}) = (f, \tilde{f})_{L^2[0,1]} + g_1 \cdot \tilde{g}_1 + g_2 \cdot \tilde{g}_2$ and the norm

$$\|\boldsymbol{f}\| = \sqrt{\|f\|^2_{L^2[0,1]} + |g_1|^2 + |g_2|^2}$$

for the Hilbert space $L^2[0,1] \times \mathbb{R}^2$. Here $\boldsymbol{f} = (f, g_1, g_2)^\top \in L^2[0,1] \times \mathbb{R}^2$ denotes the right hand side of the problem. Let us note that the vectorial operator $\boldsymbol{L}$ has the following properties.

Theorem 1. *The operator $\boldsymbol{L} : H^2[0,1] \to L^2[0,1] \times \mathbb{R}^2$ is continuous and linear with the domain $D(\boldsymbol{L}) = H^2[0,1]$ and the closed range $R(\boldsymbol{L})$.*

Proof. The proof is obtained almost the same as Theorem 1 is proved in [2]. □

If the problem has the unique solution ($\gamma\xi^2 \neq 2$), its range is coincident with the entire space $L^2[0,1] \times \mathbb{R}^2$. For the problem without the unique solution ($\gamma\xi^2 = 2$), we obtain the following range representation.

Lemma 1. *If $\gamma\xi^2 = 2$, the range of the operator $\boldsymbol{L}$ is of the form*

$$R(\boldsymbol{L}) = \left\{ \left(f, g_1, (1 - \gamma\xi)g_1 - \gamma \int_0^1\!\!\int_0^\xi G^{cl}(x,y) f(y)\, dx\, dy \right)^\top, \, g_1 \in \mathbb{R},\ f \in L^2[0,1] \right\}.$$

Proof. The general solution of the equation $-u'' = f$ is given by

$$u = c_1 + c_2 x + \int_0^1 G^{cl}(x,y) f(y)\, dy, \quad c_1, c_2 \in \mathbb{R}.$$

Substituting it into nonlocal conditions, we get $c_1 = g_1$ and

$$\left(1 - \gamma \frac{\xi^2}{2}\right) c_2 = (\gamma\xi - 1) g_1 + \gamma \int_0^1 \int_0^\xi G^{cl}(x,y) f(y)\, dx\, dy + g_2.$$

Since $\gamma\xi^2 = 2$, we solve $g_2 = (1 - \gamma\xi)g_1 - \gamma \int_0^1 \int_0^\xi G^{cl}(x,y) f(y)\, dx\, dy$. Thus, vectorial functions of the form $\boldsymbol{f} = (f, g_1, g_2)^\top$, with the obtained g_2 expression via arbitrary g_1 and f, represent the range. □

According to [1], properties of $\boldsymbol{L}$ implies the closeness of $N(\boldsymbol{L}^*)$, where $\boldsymbol{L}^*$: $L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]$ is the adjoint operator of $\boldsymbol{L}$. Then the nullspace and range theorem gives $N(\boldsymbol{L}^*) = R(\boldsymbol{L})^\perp$, which representation can also be derived in the following form.

Corollary 1. *If $\gamma\xi^2 = 2$, then we have*

$$N(\boldsymbol{L}^*) = \left\{ c \cdot \boldsymbol{w} : \ \boldsymbol{w} = \left(\gamma \int_0^\xi G^{cl}(t,x)\, dt, \gamma\xi - 1, 1 \right)^\top, \ c \in \mathbb{R} \right\}.$$

Proof. We have the orthogonality condition $(\boldsymbol{f}, \tilde{\boldsymbol{f}}) = 0$ for all $\boldsymbol{f} \in R(\boldsymbol{L})$ and $\tilde{\boldsymbol{f}} = (\tilde{f}, \tilde{g}_1, \tilde{g}_2)^\top \in R(\boldsymbol{L})^\perp = N(\boldsymbol{L}^*)$, that is given in the explicit form

$$\int_0^1 \left(\tilde{f}(y) - \tilde{g}_2\gamma \int_0^\xi G^{cl}(x,y)\,dx\right) f(y)\,dy + g_1 \cdot \big(\tilde{g}_1 + (1-\gamma\xi)\tilde{g}_2\big) = 0$$

with arbitrary $g_1 \in \mathbb{R}$ and $f \in L^2[0,1]$. Since g_1 and f obtain values independently, we take $g_1 = 0$, afterwards $f = 0$ and get two conditions $\tilde{f}(y) - \tilde{g}_2\gamma \int_0^\xi G^{cl}(x,y)\,dx = 0$ and $\tilde{g}_1 + (1-\gamma\xi)\tilde{g}_2 = 0$. From here we obtain $\tilde{f}(x) = \tilde{g}_2\gamma \int_0^\xi G^{cl}(t,x)\,dt$ and $\tilde{g}_1 = \tilde{g}_2(\gamma\xi - 1)$. Thus, the nullspace $N(\boldsymbol{L}^*)$ is of the form $\tilde{\boldsymbol{f}} = (\tilde{g}_2\gamma \int_0^\xi G^{cl}(t,x)\,dt; \tilde{g}_2(\gamma\xi-1); \tilde{g}_2)^\top$ or $\tilde{\boldsymbol{f}} = \tilde{g}_2(\gamma \int_0^\xi G^{cl}(t,x)\,dt; (\gamma\xi-1); 1)^\top$ with an arbitrary $\tilde{g}_2 \in \mathbb{R}$. □

From the Fredholm alternative theorem, we obtain the solvability condition for the problem without the uniqueness.

Corollary 2. *The problem* (1)–(2) *with* $\gamma\xi^2 = 2$ *is solvable if and only if the condition is valid:*

$$(\gamma\xi - 1)g_1 + g_2 + \gamma \int_0^1 \int_0^\xi G^{cl}(x,y) f(y)\,dx\,dy = 0.$$

3 Existence and representation of the minimizer

According to [1] and Theorem 1, the operator $\boldsymbol{L}$ has the *Moore–Penrose inverse* $\boldsymbol{L}^\dagger : L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]$. It is the unique solution to the four operator equations

$$\boldsymbol{L}\boldsymbol{L}^\dagger\boldsymbol{L} = \boldsymbol{L}, \qquad \boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{L}^\dagger = \boldsymbol{L}^\dagger, \qquad (\boldsymbol{L}\boldsymbol{L}^\dagger)^* = \boldsymbol{L}\boldsymbol{L}^\dagger, \qquad (\boldsymbol{L}^\dagger\boldsymbol{L})^* = \boldsymbol{L}^\dagger\boldsymbol{L}.$$

Moreover, the Moore–Penrose inverse describes the desired *best fit solution*

$$u^o = \boldsymbol{L}^\dagger \boldsymbol{f},$$

which is also known as *the minimum norm least squares solution* or the *best approximate solution*. Let us note that the function u^o has the minimum $H^2[0,1]$ norm among all minimizers u^g of the norm of the residual

$$\|\boldsymbol{L}u^g - \boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^2} = \min_{u\in H^2[0,1]} \|\boldsymbol{L}u - \boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^2}, \tag{5}$$

that is, $\|u^o\|_{H^2[0,1]} < \|u^g\|_{H^2[0,1]}$ for all $u^g \neq u^o$.

The minimizer u^o to the problem $\boldsymbol{L}u = \boldsymbol{f}$ (which may be consistent or inconsistent) is always the minimizer to the consistent problem $\boldsymbol{L}u = \mathbf{P}_{R(\boldsymbol{L})}\boldsymbol{f}$ [1] (here and further P_S denotes the orthogonal projector onto a subspace S). Let us solve it! First, we shortly denote the right hand side and calculate it as below

$$\tilde{\boldsymbol{f}} := \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{P}_{R(\boldsymbol{L})^\perp}\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{P}_{N(\boldsymbol{L}^*)}\boldsymbol{f} = \boldsymbol{f} - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}(\boldsymbol{w}, \boldsymbol{f})$$
$$= \begin{pmatrix} f(x) \\ g_1 \\ g_2 \end{pmatrix} - \frac{\gamma \int_0^1 \int_0^\xi G^{cl}(t,y) f(y)\,dt\,dy + g_1(\gamma\xi - 1) + g_2}{\gamma^2 \int_0^1 \big(\int_0^\xi G^{\mathrm{cl}}(t,y)\,dt\big)^2 dy + (\gamma\xi-1)^2 + 1} \begin{pmatrix} \gamma \int_0^\xi G^{\mathrm{cl}}(t,x)\,dt \\ \gamma\xi - 1 \\ 1 \end{pmatrix}.$$

The problem $\boldsymbol{L}u = \tilde{\boldsymbol{f}}$ is of the form (1)–(2) with the above calculated right hand side $\tilde{\boldsymbol{f}} = (\tilde{f}, \tilde{g}_1, \tilde{g}_2)^\top$ instead of $\boldsymbol{f} = (f, g_1, g_2)^\top$. Thus, we take the general solution of the differential equation $-u'' = \tilde{f}$, that is,

$$u = c_1 + c_2 x + \int_0^1 G^{cl}(x,y)\tilde{f}(y)\,dy, \quad c_1, c_2 \in \mathbb{R}.$$

Substituting it into the initial condition $u(0) = \tilde{g}_1$, we find $c_1 = \tilde{g}_1$. Let us note that the integral condition $u(1) = \gamma \int_0^\xi u(x)\,dx + \tilde{g}_2$ is satisfied trivially because we have the solvable problem without the uniqueness ($\gamma\xi^2 = 2$). Thus, we obtain the general solution of the form

$$u^g = \tilde{g}_1 + cx + \int_0^1 G^{cl}(x,y)\tilde{f}(y)\,dy$$

with one arbitrary constant $c \in \mathbb{R}$.

According to [1], we can find the minimizer from the formula $u^o = P_{N(\boldsymbol{L})^\perp} u^g$, that is also $u^o = u^g - P_{N(\boldsymbol{L})} u^g$. Let us note that $N(\boldsymbol{L}) = \{cx,\ c \in \mathbb{R}\}$ and, thus, $P_{N(\boldsymbol{L})}(cx) = cx$. Now we calculate

$$P_{N(\boldsymbol{L})} 1 = \frac{(x,1)_{H^2[0,1]}}{\|x\|^2_{H^2[0,1]}} x = \frac{3}{8} x$$

and obtain $P_{N(\boldsymbol{L})}\tilde{g}_1 = \tilde{g}_1 \cdot P_{N(\boldsymbol{L})} 1 = \tilde{g}_1 \cdot 3x/8$. Similarly, we calculate

$$\begin{aligned} P_{N(\boldsymbol{L})}\left(\int_0^1 G^{cl}(x,y)\tilde{f}(y)\,dy\right) &= \frac{(t, \int_0^1 G^{\mathrm{cl}}(t,y)\tilde{f}(y)\,dy)_{H^2[0,1]}}{\|t\|^2_{H^2[0,1]}} x \\ &= \int_0^1 \frac{(t, G^{\mathrm{cl}}(t,y))_{H^1[0,1]}}{\|t\|^2_{H^2[0,1]}} x \cdot \tilde{f}(y)\,dy \\ &= \int_0^1 P_{N(\boldsymbol{L})} G^{cl}(x,y)\tilde{f}(y)\,dy, \end{aligned}$$

with the kernel $P_{N(\boldsymbol{L})} G^{cl}(x,y) = \frac{1}{8} \cdot xy(1-y^2)$. Here t denotes the integration variable. Then the formula $u^o = u^g - P_{N(\boldsymbol{L})} u^g$ gives

$$u^o = \left(1 - \frac{3}{8}x\right)\tilde{g}_1 + \int_0^1 G^{cl}(x,y)\tilde{f}(y)\,dy - \int_0^1 P_{N(\boldsymbol{L})} G^{cl}(x,y)\tilde{f}(y)\,dy.$$

4 Generalized Green's function

Substituting expressions of $\tilde{g}_1$ and $\tilde{f}$, we rewrite this minimizer in the form

$$u^o = g_1 \cdot v^{g,1} + g_2 \cdot v^{g,2} + \int_0^1 G^g(x,y) f(y)\,dy, \tag{6}$$

where we denoted functions

$$v^{g,1} = (\gamma\xi - 1)v^{g,2} + 1 - \frac{3}{8}x,$$

$$v^{g,2} = \frac{(1-\gamma\xi)(1-3x/8) + \gamma\int_0^1 \big(P_{N(\boldsymbol{L})}G^{cl}(x,y) - G^{cl}(x,y)\big)\int_0^\xi G^{cl}(t,y)\,dt\,dy}{\gamma^2\int_0^1\big(\int_0^\xi G^{cl}(t,y)\,dt\big)^2 dy + (\gamma\xi-1)^2 + 1},$$

$$G^g(x,y) = G^{cl}(x,y) - P_{N(\boldsymbol{L})}G^{cl}(x,y) + \gamma v^{g,2}(x)\int_0^\xi G^{cl}(t,y)\,dt.$$

Let us note that the unique solution (case $\gamma\xi^2 \neq 2$), which is given in Introduction, can also be represented in the relative form

$$u = g_1 v^1 + g_2 v^2 + \int_0^1 G(x,y)f(y)\,dy$$

with the functions

$$v^1 = \frac{2-\gamma\xi^2 + (\gamma\xi-1)x}{2-\gamma\xi^2} \quad (\text{or} \quad v^1 = (\gamma\xi-1)v^2 + 1), \qquad v^2 = \frac{2}{2-\gamma\xi^2}x,$$

$$G(x,y) = G^{cl}(x,y) + \gamma v^2(x)\int_0^\xi G^{cl}(t,y)\,dy.$$

These functions v^1, v^2 are called the *biorthogonal fundamental system* and $G(x,y)$ is called the *Green's function* for the problem (1)–(2) (see [3]). Thus, because of the similarity, we call the functions $v^{g,1}$, $v^{g,2}$ – the *generalized biorthogonal fundamental system* and $G^g(x,y)$ – the *generalized Green's function* for the problem (1)–(2) without the uniqueness (case $\gamma\xi^2 = 2$).

As in the paper [2], let us calculate the error, which is made by the best fit solution. It is the minimum norm of the residual (5) and is equal to

$$\text{min. res.} = \frac{|(\boldsymbol{w},\boldsymbol{f})|}{\|\boldsymbol{w}\|} = \frac{\big|\gamma\int_0^1\int_0^\xi G^{cl}(t,y)f(y)\,dt\,dy + g_1(\gamma\xi-1) + g_2\big|}{\sqrt{\gamma^2\int_0^1\big(\int_0^\xi G^{cl}(t,y)\,dt\big)^2 dy + (\gamma\xi-1)^2 + 1}}. \qquad (7)$$

Let us note that it vanishes if the problem is consistent (condition of Corollary 2 is fulfilled). In this case $\boldsymbol{f} \in R(\boldsymbol{L})$, $\boldsymbol{w} \in N(\boldsymbol{L}^*) = R(\boldsymbol{L})^\perp$, what gives $(\boldsymbol{w},\boldsymbol{f}) = 0$.

Example 1. Let us now consider the particular problem (1)–(2) with $f(x) = 0$, $x \in [0,1]$, $g_1 = 0$, $g_2 = 1$, $\gamma = 8$ and $\xi = 1/2$, that is,

$$-u''(x) = 0, \quad x \in [0,1], \qquad u(0) = 0, \quad u(1) = 8\int_0^{1/2} u(x)\,dx + 1.$$

We can directly verify that this problem has no solutions. Moreover, the formula (6) says that in this case $u^o = v^{g,2}$. Simplifying the derived $v^{g,2}$ expression via Green's function (3) above, we find the minimizer

$$v^{g,2} = \frac{135}{137}x - \frac{360}{137} - \frac{1}{96\cdot 137}\begin{cases} 32x^4 - 48x^3 - 37x, & x \leqslant 1/2, \\ 8x^3 - 48x^2 - 21x + 2, & x \geqslant 1/2. \end{cases}$$

According to (7), the norm of the minimal residual is equal to $\sqrt{120/137}$. Substituting the $v^{g,2}$ expression above, we find

$$v^{g,1} = \frac{2829}{1096}x - \frac{943}{137} - \frac{1}{32\cdot 137}\begin{cases} 32x^4 - 48x^3 - 37x, & x \leqslant 1/2,\\ 8x^3 - 48x^2 - 21x + 2, & x \geqslant 1/2,\end{cases}$$

and the generalized Green's function

$$G^g(x,y) = \begin{cases} y(1-x), & y \leqslant x,\\ x(1-y), & y \geqslant x,\end{cases} \qquad - \frac{1}{8}xy(1-y^2) + v^{g,2}(x)\begin{cases} 3y - 4y^2, & y \leqslant 1/2,\\ 1-y, & y \geqslant 1/2.\end{cases}$$

Using these functions $v^{g,1}$, $v^{g,2}$ and $G^g(x,y)$ in the formula (6), we can always find the minimizer of the problem (1)–(2) with $\gamma = 8$, $\xi = 1/2$ and every values of right hand sides f, g_1, g_2. The error, made by this best fit solution, is equal to

$$\text{min. res.} = \frac{\sqrt{120}}{\sqrt{137}}\left|3g_1 + g_2 + \int_0^{1/2}(3y-4y^2)f(y)\,dy + \int_{1/2}^1 (1-y)f(y)\,dy\right|.$$

According to Corollary 2, the problem ($\gamma = 8$ and $\xi = 1/2$) is solvable if and only if the following condition is valid

$$3g_1 + g_2 + \int_0^{1/2}(3y-4y^2)f(y)\,dy + \int_{1/2}^1 (1-y)f(y)\,dy = 0,$$

what gives the vanishing residual above in the solvable case.

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REZIUMĖ

Antrosios eilės diferencialinio uždavinio su integraline sąlyga minimalusis sprendinys

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Šiame darbe nagrinėsime antrosios eilės diferencialinio uždavinio su viena pradine ir kita integraline sąlygomis geriausiai tinkantį (mažiausių kvadratų) sprendinį. Gausime šios minimizuojančios funkcijos išraišką ir pateiksime pavyzdį.

Raktiniai žodžiai: integralinė sąlyga, mažiausių kvadratų sprendinys, Moore–Penrose atvirkštinis atvaizdis, minimizuojanti funkcija.