

Intellectual need for mathematical knowledge

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Abstract. Harel's [2] notion of *intellectual need* is refined by employing Davis' [1] findings about interesting propositions in social sciences. A few hypothetical examples of how this revised definition might aid in planning mathematics lessons which provide meaningfulness for the students are presented.

Keywords: intellectual need, mathematics education, meaningfulness.

Introduction

Most of the material that is taught in a mathematics classroom is perceived as meaningless by the students. Kaput [3] has nicely put it in his work:

Few now deny that school mathematics as experienced by most students is compartmentalized into meaningless pieces that are isolated from one another and from the students' wider world. [...] This **experienced meaningfulness** of school mathematics devastates the motivation to learn or use mathematics and is entirely incompatible with a view of mathematics as a tool of personal insight and problem solving.

30 years have passed since Kaput's paper but little has changed. Students often do not see the reason why new material is being introduced, e.g. when multiplication of fractions is introduced, students naturally ask the teacher "Why would I ever need to multiply fractions?" No satisfactory answer is given and so "most students, even those who are eager to succeed in school, feel intellectually aimless in mathematics [...]" [2]

The notion of intellectual need

Harel [2] distinguishes two kinds of needs that children are trying to fulfill when learning mathematics and terms them affective needs and intellectual needs.

He describes *affective needs* as the needs that are strongly linked to social and cultural values and conventions. It is argued that such needs can originate from (1)

external expectation (e.g. teachers, parents, society); (2) aspiration to self-advance (e.g. improving one's economic condition); or (3) desire to advance societal causes (e.g. solving a medical problem using mathematics). Harel stresses that such needs have profound importance in education, but also points out that they do not suffice.

It is also pointed out that there exists a different kind of need that can and ought to be fulfilled when learning mathematics – intellectual need. Harel defines intellectual need as a subjective state experienced by an individual where prior to acquiring a piece of knowledge K , an individual I faces a problematic situation S , which is unsolvable without K . In such a situation, I is said to have an *intellectual need* for K . The author points out that intellectual need has to do with the *epistemology of the subject* and is less dependent or influenced by social or cultural values.

Harel's definition is important as it addresses a key problem in mathematics education, namely that of the lack of meaningfulness. However, it is difficult to use for practical purposes, as the term *problematic situation* is not explicitly defined. He only clarifies that for the term *problem solving* he uses the definition provided in [4], namely engagement in a problem for which the solution method is not known in advance. However, that does not help.

Consider the following situation: a teacher asks the students to prove a theorem. Is that a problematic situation? Based on the way Harel defines problem solving, we could guess that the answer is positive, since we can assume that the students do not know the proof of the theorem and hence that is a situation to which a solution method is not known in advance. Then according to the definition of intellectual need, the students would feel an intellectual need for the proof of that theorem. But that means the students would feel an intellectual need for every proof that they do not know in advance. Clearly though every teacher would confidently assure us that simply asking students to prove a theorem does not cause any intellectual need for the students.

So it necessarily follows that asking students to prove a theorem is not a problematic situation. We naturally begin to wonder why not and need to understand what properties make a situation problematic.

I am going to suggest a way to improve Harel's definition of intellectual need by connecting the ideas of M. S. Davis' paper [1] in which he explains what makes a proposition interesting.

Revising Harel's definition

Davis [1] "examined a large number of social and especially sociological propositions which have been considered interesting in the hope of isolating the common element of 'interest' in all of them [...]" He considers a proposition to be interesting if it has been in "wide circulation".

Davis found that what makes a proposition interesting is its ability to deny the reader's *assumption ground* and usually has the following structure: "what appears to be X is in actuality non- X ". For example Freud's proposition that love and hate can be compatible is interesting because it is usually assumed that love and hate are incompatible. Another example of an interesting proposition would be a statement "Jails make people criminals". It is interesting because it denies our intuition for it is normally assumed that the purpose of jails is to make people not be criminals.

Table 1. Truth of certain statements in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^+$, here $m, n, k \in \mathbb{N}$ and $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$.

Statement	Always true in \mathbb{Z} ?	Always true in \mathbb{Q}^+ ?
$m + n = k \implies k > m, k > n$	No	Yes
$mn = k \implies k \geq m, k \geq n$	No	No
$m - n = k \implies k < m$	No	Yes
$m : n = k \implies k \leq m$	No	No
$(m - n) \in \mathbb{N} \iff m > n$	No	Yes
$(m : n) \in \mathbb{N} \iff \exists l \in \mathbb{N} \mid m = nl$	Yes	No
$mn = \underbrace{n + n + \dots + n}_{m \text{ times}}$	No	No

Employing Davis’ ideas I will now attempt to improve Harel’s definition of intellectual need so as to further clarify what it means for a situation to be problematic. I define a situation to be problematic if it contradicts the assumption ground of an individual. Let me introduce some notation to make the new definition more concise.

Let I denote an individual, and $A(I)$ denote their *assumption ground*, i.e. the set of statements I believes to be true. Suppose after a situation S , there appears a statement in $A(I)$ such that it is contradictory to another statement in $A(I)$. In such a case S is called a *problematic situation* and I is said to feel an *intellectual need* for $K(S)$ – a piece of knowledge which would resolve the contradiction between the two contradicting elements of $A(I)$.

Putting ideas into practice

Suppose we want to teach students material K . Given the definition stated above, the general model of how to make the students experience an intellectual need for K is then as follows:

1. Recognize key aspects and ideas about K ;
2. Identify the likely assumption ground of a given set of students related to key aspects and ideas from the previous step;
3. Figure out how the assumption ground of the students can be denied or contradicted by the key aspects and ideas about K ;
4. Model a situation S in which the students would become aware of the contradiction;
5. Model your lesson so that students are able to see how K resolves S and become aware of that resolution.

Extending a number system

The number system is extended a few times in school. Until students learn of integers, their intuition and thus assumption ground about numbers is limited to ideas about natural numbers. Until we introduce rational numbers, they intue that all numbers behave like natural numbers or integers. We can use such assumption ground and try to contradict it. In Table 1 I suggest some statements which are true for natural

numbers but not necessarily true for integers or rational numbers. Let us now attempt to use such statements in order to model classroom situations which would provoke an intellectual need for integers or rational numbers.

Does a quotient always exist? Suppose we have a group of students who believe that a quotient can only exist as a natural number. We then give these students a problem: “On Monday a group of 7 friends share 14 cookies, but on Tuesday the same group of friends share 15 cookies. How many cookies did each individual get on a single day?” Monday is easy: $14 : 7 = 2$, however Tuesday is trickier. The teacher should not accept an answer like “Everyone got 2 and 1 cookie remained” and should stress that it is possible to divide the remaining cookie into 7 equal pieces. At first the students should draw the solution in pictures and then begin to reason of how to denote their drawings using mathematical symbols. According to their belief it is impossible to divide 15 by 7, however reality shows that it is indeed possible. Clearly an intellectual need for introducing a new kind of numbers, namely, positive rational numbers is experienced.

The sum is always 10! Say a group of students believe that the sum is always greater than the addends. We set up a spreadsheet like Microsoft Excel where in cells A1 and A2 one can freely input any numbers and in A3 a formula $=A1+A2$ is written. We project this spreadsheet on a screen so the whole class can see and at first we let students play with the numbers so that they become convinced that A3 always outputs the sum correctly. The teacher then changes the font color of A2 so that it matches the background color of the cell in order for children not to see what the input of that cell is and says: “We will play a game. You choose a number which I shall input to A1. If I manage to put a number into A2 so that the sum in A3 is equal to 10, I get a point, if I fail, You get a point. Let’s see who scores more points.” The situation concludes with the teacher scoring all the points and the students coming to terms that it is possible to get a sum which is less than the addend and then feeling an intellectual need to discover the teacher’s secret, i.e. negative numbers.

More generally new classes of numbers are introduced so that more equations become solvable:

1. Negative numbers are introduced so that $a + x = b$ is always solvable, when $a, b \in \mathbb{N}$;
2. Rational numbers are introduced so that $ax = b$ is always solvable, when $a, b \in \mathbb{Z}$;
3. Roots (and logarithms) are introduced so that $x^a = b$ (and $a^x = b$) is always solvable, when $a, b \in \mathbb{N}$;
4. Complex numbers are introduced so that $x^a = b$ is always solvable, when $a \in \mathbb{N}, b \in \mathbb{Z}$.

Until learning about the existence of these numbers, the students believe that the equations in the above list are unsolvable. By using a similar technique described in the above paragraph, the teacher can show that he or she is actually able to solve such equations. The students would then experience a contradiction between their prior belief that the equation $f(x) = b$ is unsolvable and the realization that the teacher

has just solved it. Hence they would experience an intellectual need for a new class of numbers.

Responding to students ideas

A usual situation in the classroom is that a student presents his idea of which a part is wrong. Let's explore how we can use these partly wrong ideas to cause an intellectual need to be felt by the student for more rigorous understanding.

Limits. Let us consider an example given by Harel in [2]. Suppose a student says: " $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, because the larger x gets, the closer $\frac{1}{x}$ is to 0". The author then suggests to respond along the lines of the following statement "But then according to your own explanation $\lim_{x \rightarrow \infty} \frac{1}{x} = -1$, because the larger x gets, the closer $\frac{1}{x}$ is to -1 ". Such a response creates a contradiction between the student's belief that they understand why said limit is equal to 0 and the realization that actually they do not understand the real reason behind the equation. Hence they experience an intellectual need to find the real reason for the equation.

Such an intellectual need would not be experienced if the teacher would respond in a more usual manner: "Not precise enough, try using the language of ϵ and δ ". In such a case no contradiction would be created, the student's belief of understanding the true reason why $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ would not be shaken and no intellectual need would be experienced. The student would probably mutter something along the lines of "Why is the teacher so picky? There's nothing wrong with my explanation! Why does mathematics have to be so weird?"

Fraction addition. A classic mistake made when students are learning fraction addition is for them to add the fractions thus: $\frac{1}{3} + \frac{2}{5} = \frac{1+2}{3+5} = \frac{3}{8}$. We assume that if the student uses such a method to add fractions, he or she believes it to be true, at least partly. The teacher could then show such equations:

$$\frac{1}{1} + \frac{1}{1} = \frac{1+1}{1+1} = \frac{2}{2} = 1 \quad \text{and} \quad \frac{1}{1} + \frac{1}{1} = 1 + 1 = 2,$$

and ask the student to explain what happened. The student would quickly realize that his approach of adding fractions leads to a contradiction and hence would understand that his method is actually incorrect. That denies his or her prior belief to be able to correctly add fractions. Hence the student experiences an intellectual need for the true method of adding fractions.

Conclusions

I have demonstrated that it is possible to model classroom situations in which students experience an intellectual need for a certain piece of mathematical knowledge. I hypothesize that introducing material K only when students have experienced an intellectual need for it would result in students experiencing school mathematics in a more meaningful way and would help them appreciate mathematics as a tool of "personal insight and problem solving" [3].

However, a lot of further research is needed. My hypothesis needs to be tested and if supported by evidence, a database needs to be created which ideally for every knowledge piece of school mathematics K would describe: (1) an assumption ground that needs to be possessed by the students in order for K 's introduction to be feasible; and (2) hypothetical classroom situations that by utilizing a given assumption ground would help elicit an intellectual need for K . The database naturally would need to be in constant empirical testing and refinement.

References

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REZIUMĖ

Matematinių žinių intelektualinis poreikis

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Harel [2] *intelektinio poreikio* sąvoka atnaujinama panaudojant Davis [1] įžvalgas apie įdomias tezes socialiniuose moksluose. Pateikiama keletas hipotetinių pavyzdžių, kaip išnaudojant patikslintą intelektualinio poreikio sąvoką matematikos pamokose galima suteikti moksleiviams prasmingumo pojūtį.

Raktiniai žodžiai: intelektualinis poreikis, matematikos švietimas, prasmingumas.