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Modelling of an age-structured population dynamics taking into account a discrete set of offspring

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Abstract. A model for description of an age-structured population dynamics taking into account a discrete set of offspring, their care, and spatial diffusion in two-dimensional space is studied numerically. The model consists of a coupled system of integro-partial differential equations. Some results are illustrated by figures.

Keywords: age-structured population, child care, spatial diffusion.

Introduction

The child care phenomenon is natural for many species of mammals and birds. It forms the main difference between the behaviour of populations that care for their young offspring and those without maternal (or parental) duties. Mammals and birds produce a small number of offspring then feed, warm, protect from predators and starvation, and train them to find food for themselves. If one of these natural duties is not realized, young offspring die and the population vanishes. For many species of mammals and birds both parents care for their offspring. It is also known that for some species of mammals (e.g., beer, whale, and panther) only a female provides care of her young offspring.

The one-sex Sharpe–Lotka–McKendrick–von Foerster [6, 4, 11, 2] model and two-sex Fredrickson–Hoppensteadt–Staroverov [1, 3, 10] permanent pairs model are known in mathematical biology. However, all these models do not treat the child care phenomenon and cannot be used to describe the evolution of populations that care for their offspring. They have to be applied only for the dynamics description of populations that do not take care of their young offspring, e.g. some species of fishes, reptile, and amphibia.

In recent years, three models for an age-structured wild populations based on the discrete set of newborns were proposed and examined analytically [7, 8]. Numerical results of solving of a one-sex age-child-structured population model without and with spatial diffusion in $\mathbf{R}^1$ are given in [5] taking into account a discrete set of offspring and their care.

The aim of this paper is numerical solving of this model in $\mathbf{R}^2$ by using the alternating directions method.

The paper is organized as follows. In Section 2, we describe the model. Some numerical results are demonstrated Section 3.

1 Notation

We use the notation of paper [9].

$\mathbf{R}^2$: the Euclidean space of dimension 2 with $x = (x_1, x_2)$,

κ: the diffusion modulus,

$(0, T)$ and (T_1, T_3) $(T < T_1 < T_3)$: the child care and reproductive age intervals, respectively,

$u(t, \tau_1, x)$: the age-space-density of individuals aged τ_1 at time t at the position x who are of juvenile ($\tau_1 \in (T, T_1)$), single ($\tau_1 \in (T_1, T_3)$), or post-reproductive ($\tau_1 > T_3$) age,

$u_k(t, \tau_1, \tau_2, x)$: the age-space-density of individuals aged τ_1 at time t at the position x who take care of their k offspring aged τ_2 at the same time,

$\nu(t, \tau_1, x)$: the natural death rate of individuals aged τ_1 at time t at the position x who are of juvenile or adult age,

$\nu_k(t, \tau_1, \tau_2, x)$: the natural death rate of individuals aged τ_1 at time t at the position x who take care of their k offspring aged τ_2,

$\nu_{ks}(t, \tau_1, \tau_2, x)$: the natural death rate of k-s young offspring aged τ_2 at time t at the position x whose mother is aged τ_1 at the same time,

$\alpha_k(t, \tau_1, x)dt$: the probability to produce k offspring in the time interval $[t, t + dt]$ at the location x for an individual aged τ_1,

N: sum of spatial densities of juvenile and adult individuals,

$\rho(N)$: the death rate conditioned by ecological causes (overcrowding of the population), $\rho(0) = 0$,

$u_0(\tau_1, x)$, $u_{k0}(\tau_1, \tau_2, x)$: the initial age distributions,

$[u|_{\tau_1=\tau}]$: the jump discontinuity of u at the point $\tau_1 = \tau$,

$\alpha = \sum_{k=1}^{n} \alpha_k$, $\gamma_1(\tau_1) = \max(0, \tau_1 - T_3)$, $\gamma_2(\tau_1) = \min(\tau_1 - T_1, T)$,

$\tilde{\nu}_k = \nu_k + \sum_{s=0}^{k-1} \nu_{ks}$,

$T_2 = T_1 + T$: the minimal age of an individual finishing care of offspring of the first generation,

$T_4 = T_3 + T$: the maximal age of an individual finishing care of offspring of the last generation,

$\sigma_1 = (T_1, T_3)$, $\sigma_2 = (T_1, T_4)$, $\sigma_3 = (T_2, T_4)$,

$\sigma_1{}^* = (T, \infty) \setminus \sigma_1$, $\sigma_2{}^* = (T, \infty) \setminus \sigma_2$, $\sigma_3{}^* = (T, \infty) \setminus \sigma_3$,

$Q = \{(\tau_1, \tau_2) : \tau_1 \in (T_1 + \tau_2, T_3 + \tau_2), \tau_2 \in (0, T)\}$.

In what follows, κ, T, T_1, and T_3 are assumed to be positive constants.

2 Model

To describe the population dynamics including the spatial diffusion in the $\Omega \subset \mathbf{R}^2$ with the extremely inhospitable $\partial\Omega$ and constant diffusion modulus κ we use the model [9]

$$\begin{cases} \partial_t u + \partial_{\tau_1} u + (\nu + \rho(N))u - \kappa \Delta u = -\begin{cases} 0, & \tau_1 \in \sigma_1^*, \\ \alpha u, & \tau_1 \in \sigma_1 \end{cases} \\ \quad + \begin{cases} 0, & \tau_1 \in \sigma_2^*, \\ \int_{\gamma_1(\tau_1)}^{\gamma_2(\tau_1)} \sum_{k=1}^{n} \nu_{k0} u_k \, d\tau_2, & \tau_1 \in \sigma_2 \end{cases} \\ \quad + \begin{cases} 0, & \tau_1 \in \sigma_3 *, \\ \sum_{k=1}^{n} u_k|_{\tau_2 = T}, & \tau_1 \in \sigma_3, \end{cases} \qquad t > 0,\ x \in \Omega, \\ u|_{\tau_1 = T} = \int_{\sigma_3} \sum_{k=1}^{n} k u_k|_{\tau_2 = T} \, d\tau_1, \\ u|_{t=0} = u_0, \quad [u|_{\tau_1 = \tau}] = 0, \quad \tau = T_1, T_2, T_3, T_4, \quad u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

$$\begin{cases} \partial_t u_k + \partial_{\tau_1} u_k + \partial_{\tau_2} u_k + (\tilde{\nu}_k + \rho(N))u_k - \kappa \Delta u_k \\ \quad = \begin{cases} 0, & k = n, \\ \sum_{s=k+1}^{n} \nu_{sk} u_s, & 1 \leqslant k \leqslant n-1, \end{cases} \qquad (\tau_1, \tau_2) \in Q,\ t > 0,\ x \in \Omega, \\ u_k|_{\tau_2 = 0} = \alpha_k u, \quad u_k|_{t=0} = u_{k0}, \quad u_k|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

$$N = \int_T^\infty u d\tau_1 + \int_0^T d\tau_2 \int_{T_1 + \tau_2}^{T_3 + \tau_2} \sum_{k=1}^{n} u_k \, d\tau_1 \tag{3}$$

subject to the compatibility conditions

$$\begin{cases} u_0|_{\tau_1 = T} = \int_{\sigma_3} \sum_{k=1}^{n} k u_{k0}|_{\tau_2 = T} d\tau_1, \quad u_0|_{\partial\Omega} = 0, \\ [u_0|_{\tau_1 = \tau}] = 0, \quad \tau = T_1, T_2, T_3, T_4, \\ u_{k0}|_{\tau_2 = 0} = (\alpha_k)|_{t=0} u_0, \quad , u_{k0}|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

Here $\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2}$, ∂_t, ∂_{τ_k} and $\partial_{x_s x_s}$ denote partial derivatives, while the number n is the biologically possible maximal number of the newborns of the same generation produced by an individual. The first term on the right-hand side in Eq. (1) means the part of individuals who produce offspring, the second and third terms describe the part of individuals whose all young offspring die and who finish child care, respectively. The transition term $\sum_{s=0}^{k-1} \nu_{ks} u_k$ on the left-hand side in Eq. (2) describes the part of individuals aged τ_1 at time t who take care of k young offspring and whose at least one young offspring dies. Similarly, the term on the right-hand side in this equation describes a part of individuals aged τ_1 at time t who take care of more than k, $1 \leqslant k \leqslant n-1$, young offspring aged τ_2 whose number after the death of the other offspring becomes equal to k. The condition $[u|_{\tau_1 = \tau}] = 0$, $\tau = T_1, T_2, T_3$, and T_4, means that the function u must be continuous at the point, $\tau_1 = \tau$, of the discontinuity of the right-hand side of Eq. (1).

3 Numerical results

Numerical schemes for solving of system (1)–(4) in the case of one-dimensional Ω are given in [5]. In the present section, we consider system (1)–(4) in the two-dimensional case of Ω and demonstrate some numerical results. We consider the case $n = 3$. This correspond to species such as *Felis yagouarundi* (2–3 children), *Pseudocheirus peregrinus* (1–3 children), *Tremarctos ornatus* (1–3 children) and *Arctictis binturong* (1–3 children). In all calculations we use the following model vital rates

$$\begin{cases} \nu(\tau_1) = \mu_1\tau_1^q + \mu_2, & q > 1, \\ \nu_k(\tau_1,\tau_2) = \mu_{k1}\tau_1^{q_k} + \mu_{k2}, & q_k > 1, \\ \nu_{ks}(\tau_1,\tau_2) = \mu_{ks1}|\tau_2 - \tau_0|^q + \mu_{ks2}, & \tau_0 < T, \\ \alpha_k(\tau_1) = \alpha_{k1}\exp\{-|(\tau_1 - (T_3+T_1)/2)|^{q_0}/\alpha_{k2}\}, & q_0 > 1, \\ \rho(N) = \rho_0 N^{\xi_1}, & \xi_1 > 0 \end{cases} \tag{5}$$

and initial functions

$$\begin{cases} u_0(\tau_1,x) = F_0(x)\beta_3(p(x)\tau_1 + \beta_2)\exp\{-\beta_1 p_\star(x)\tau_1\}, \\ u_{k0}(\tau_1,\tau_2,x) = \alpha_k(\tau_1-\tau_2)u_0(\tau_1-\tau_2,x)\widetilde{U}_k(\tau_2), \quad x \in [0;1]\times[0;1]. \end{cases} \tag{6}$$

Here

$$\tilde{U}_k(\tau_2) = 1 + \frac{\tau_2}{T}\big(\tilde{U}_k(T) - 1\big), \qquad p(x) = 1 + p_1 x_1^{\xi_2}(1-x_1)^{\xi_2}x_2^{\xi_2}(1-x_2)^{\xi_2},$$
$$p_\star(x) = 1 + p_2 x_1^{\xi_2}(1-x_1)^{\xi_2}x_2^{\xi_2}(1-x_2)^{\xi_2},$$
$$F_0(x) = Ax_1^{\xi_3}(1-x_1)^{\xi_3}x_2^{\xi_3}(1-x_2)^{\xi_3}, \quad x = (x_1,x_2).$$

Using compatibility conditions (4) we determine function

$$\beta_2(x) = -p(x)T + p(x)\int_{T_1}^{T_3}\sum_{k=1}^{3}\tilde{U}_k(T)k\alpha_k(\tau_1)(\tau_1 - T)\exp\big\{-p_\star(x)\beta_1\tau_1\big\}\,d\tau_1$$
$$\times\bigg\{\exp\big\{-p_\star(x)\beta_1 T\big\} - \int_{T_1}^{T_3}\sum_{k=1}^{3}\tilde{U}_k(T)k\alpha_k(\tau_1)\exp\big\{-p_\star(x)(x)\beta_1\tau_1\big\}\,d\tau_1\bigg\}^{-1}. \tag{7}$$

The positive constants μ_1, μ_2, μ_{k1}, μ_{k2}, μ_{ks1}, μ_{ks2}, $q_0 > 1$, $q > 1$, $q_k > 1$, β_1, β_3, $\tilde{U}_k(T)$, p_1, p_2, ξ_1, $\xi_2 \geqslant 1$, $\xi_3 \geqslant 1$, $T < T_1 < T_3$, A, α_{k1}, α_{k2}, $\tau_0 < T$, and ρ_0 remain free.

To solve Eqs. (1)–(4) and (5)–(7) we use the method of the alternating directions which leads to the one dimensional models with respect to x_1 or x_2. To solve them we use the Crank–Nicolson scheme together with the iteration procedure.

All calculations are performed for:

$$\begin{gathered} \beta_1 = 0.55, \quad \beta_2 = 5.7, \quad \alpha_{12} = 4, \quad \alpha_{22} = 4, \quad \alpha_{32} = 4, \\ \mu_k = 0.01, \quad \mu_{2k} = \mu_{3k} = 0.001, \quad \mu_{32k} = \mu_{21k} = \mu_{10k} = 0.0012, \\ \mu_{31k} = \mu_{20k} = 0.001, \quad \mu_{30k} = 0.0008, \quad k = 1,2, \\ \tilde{U}_1 = 0.7, \quad \tilde{U}_2 = 0.6, \quad \tilde{U}_3 = 0.5, \quad A = 3, \quad \tau_0 = 0.2, \\ \xi_1 = 1.5, \quad \xi_2 = 1.5, \quad \xi_3 = 1.5, \quad q_0 = 1.5, \quad q_1 = 2, \quad q_2 = 2, \quad q = 2, \\ T = 1, \quad T_1 = 2, \quad T_2 = 3, \quad T_4 = 4, \quad T_4 = 5, \\ \alpha_{11} = 0.7, \quad \alpha_{21} = 0.75, \quad \alpha_{31} = 0.7, \quad p_1 = 1, \quad p_2 = 0, \quad \kappa = 0.03, \quad \rho_0 = 0.001. \end{gathered} \tag{8}$$

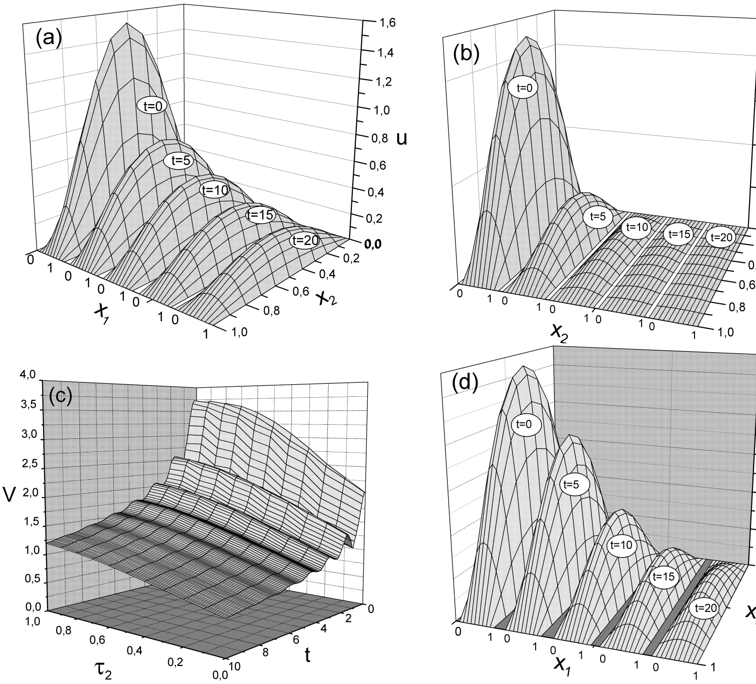

Fig. 1. Graphs of $u(t, 2, x_1, x_2)$ (a), $Z_1(t, x_1, x_2)$ (b), $V(t, \tau_2, 0.5, 0.5)$ (c), and $N(t, x_1, x_2)$ (d) for values of parameters values of parameters from (8).

Define

$$\begin{cases} V(t,\tau_2,x_1,x_2) = \displaystyle\int_{\tau_2+T_1}^{\tau_2+T_3} \sum_{k=1}^{3} k u_k \, d\tau_1, \\ Z_k(t,x_1,x_2) = \displaystyle\int_0^T d\tau_2 \int_{\tau_2+T_1}^{\tau_2+T_3} u_k(t,\tau_1,\tau_2,x_1,x_2)\, d\tau_1, \\ N_{tot}(t) = \displaystyle\int_0^1 dx_1 \int_0^1 N(t,x_1,x_2)\, dx_2. \end{cases} \tag{9}$$

Functions $V(t, \tau_2, x_1, x_2)$ and $Z_k(t, x_1, x_2)$ mean densities of offspring aged τ_2 at the positions (x_1, x_2) at time t and the density of individuals taking care of k offspring at time t at the location (x_1, x_2). $N_{tot}(t)$ means the total number of juveniles and adult individuals at time t. Some results of numerical calculations are illustrated in Fig. 1.

In Fig. 1 the graphs of dying population densities $u(t, 2, x_1, x_2)$ (Fig. 1(a)), $Z_1(t, x_1, x_2)$ (Fig. 1(b)), $V(t, \tau_2, 0.5, 0.5)$ (Fig. 1(c)), and $N(t, x_1, x_2)$ (Fig. 1(d)) at $t = 0, 5, 10, 15, 20$ are plotted.

We conclude the paper by summarising main results. One-sex age-structured population model [8] is examined numerically taking into account an environmental pressure, a discrete set of offspring, child care, and spatial diffusion in $\Omega \in \mathbf{R^2}$. Numerical scheme based on the method of the characteristic lines together with the iteration procedure is proposed and tested for initial functions of type (6). Numerical results show the rapid convergence of the scheme.

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REZIUMĖ

Populiacijos, turinčios amžių bei diskrečią vaikų aibę, dinamikos modeliavimas

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Darbe tiriamas populiacijos dinamikos kitimas skaitiniais metodais. Tariama, kad populiacijos individai yra tam tikro amžiaus ir difunduoja dvimačiame areale. Jie taip pat rūpinasi savo diskrečia aibe palikuonių. Modelis sudarytas iš kelių integro-diferencialinių lygčių sistemos. Pateikiami keli skaičiavimo rezultatai.

Raktiniai žodžiai: erdvinė difuzija, vaiko priežiūra, populiacija su amžiaus struktūra.