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Finding of roots of the matrix transcendental characteristic equation using Lambert W function

Irma Ivanovienė, Jonas Rimas

Department of Applied Mathematics, Kaunas University of Technology
Studentų 50, LT-51368, Kaunas
E-mail: irma.ivanoviene@yahoo.com; jonas.rimas@ktu.lt

Abstract. The method of finding roots of the matrix transcendental characteristic equation, corresponding to linear matrix differential equation with delayed argument, is analyzed. The examples of the application of the method are presented.

Keywords: differential equations, delayed arguments, Lambert W function, transcendental characteristic equation.

Introduction

Every real dynamical system characterizes by more or less expressed effect of a delay. Sometimes this effect is so insignificant that it can be neglected at making the mathematical model of the dynamical system. If the delays inherent in the system have significant influence to its characteristics, they must be evaluated, otherwise the conclusions got about the system analyzing its mathematical model can be erroneous. One of most often used mathematical models of dynamical systems with delays are the linear differential equations with delayed argument. The characteristic equations of such differential equations are transcendental and have infinite number of roots. The problem of finding of roots of the transcendental characteristic equation is one of the main problems which we meet while analyzing dynamical systems with delays. In the presented work the method of finding of roots of the matrix transcendental characteristic equation, based on the use of the Lambert W function, is considered. The method is used solving the matrix transcendental characteristic equation in which the coefficient matrices commute.

1 Formulation of the problem

Consider the following linear matrix differential equation with delayed argument, which is the mathematical model of some dynamical system with delays:

$$\begin{aligned} x'(t) + Bx(t) + Ax(t-T) = 0, \\ x(t) = \phi(t), \quad t \in [-T, 0], \end{aligned} \tag{1}$$

here $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ is the desired vector function, T is a constant delay, $\phi(t)$ is the initial vector function, A and B are the $n \times n$ real commuting numerical matrices.

We shall write the matrix transcendental characteristic equation, corresponding to the differential equation (1).

Assume that a solution of the matrix differential equation (1) is a vector function

$$x(t) = \mathrm{e}^{St}C, \tag{2}$$

where S is a $n \times n$ numerical matrix, C is nonzero numerical vector with n entries (the entries of S and C are some complex numbers). Having substituted this vector function into matrix differential equation (1), we get

$$\left(S + B + A\mathrm{e}^{-ST}\right)\mathrm{e}^{St}C = 0. \tag{3}$$

Assuming $\mathrm{e}^{St}C \neq 0$, we obtain the equation

$$S + B + A\mathrm{e}^{-ST} = 0, \tag{4}$$

which is the matrix transcendental characteristic equation, corresponding to the matrix differential equation with delayed argument (1). So, the vector function (2) will be the solution of the matrix differential equation (1), if and only if, the matrix S, which is in the expression of this function, will be the root of the matrix transcendental characteristic equation (4).

In what follows we will search the roots of the characteristic equation (4), using the method based on the application of the Lambert W function.

2 Lambert W function

The inverse function for the function

$$z = \psi(w) = we^{w} \tag{5}$$

(here z and w are complex variables) defines the Lambert W function, denoted by $W(z)$ [1]:

$$w = \psi^{-1}(z) = W(z). \tag{6}$$

From (5) and (6) follows, that every function $W(z)$, which satisfies the equality

$$W(z)\mathrm{e}^{W(z)} = z, \tag{7}$$

is a Lambert W function (if z is a matrix, then $W(z)$ is called a matrix Lambert W function).

The Lambert function $W(z)$ has infinite number of branches: $W_k(z)$, $k = 0, \pm1, \pm2 \ldots.$ The complete expressions of these branches can be found in [1]. The principal branch of the Lambert W function can be represented by the following power series:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n+1}}{n!} z^n. \tag{8}$$

One of the advantages of the Lambert W function method is that one can compute analytically values of the Lambert W function in all its branches using commands already installed in various software packages [3].

3 Solution of matrix transcendental characteristic equation

Consider matrix transcendental characteristic equation (4). Multiplying both sides of it by e^{St}, we get

$$(S+B)\mathsf{e}^{ST} = -A. \tag{9}$$

Performing further transformations, we multiply both sides of (9) by $T\mathsf{e}^{BT}$. This yields

$$(S+B)T\mathsf{e}^{ST}\mathsf{e}^{BT} = -AT\mathsf{e}^{BT}. \tag{10}$$

Writing (1) we have assumed, that matrices A and B commute. In [2] it is shown, that if A and B commute, the matrices S and B also commute and the following equality holds: $(S+B)T\mathsf{e}^{ST}\mathsf{e}^{BT} = (S+B)T\mathsf{e}^{(S+B)T}$. Taking this into account, we rewrite (10):

$$(S+B)T\mathsf{e}^{(S+B)T} = -AT\mathsf{e}^{BT}. \tag{11}$$

Denoting

$$W(-AT\mathsf{e}^{BT}) = (S+B)T \tag{12}$$

and using (11), we have

$$W(-AT\mathsf{e}^{BT})\mathsf{e}^{W(-AT\mathsf{e}^{BT})} = -AT\mathsf{e}^{BT}. \tag{13}$$

From (7) and (13) follows, that $W(-AT\mathsf{e}^{BT})$ is a value of the matrix Lambert W function $W(Z)$ at $Z = -AT\mathsf{e}^{BT}$. Solving (12) for S, we get the solution of the matrix transcendental characteristic equation (4):

$$S = \frac{1}{T}W(-AT\mathsf{e}^{BT}) - B. \tag{14}$$

Since the Lambert W function has infinite number of branches, the matrix transcendental characteristic equation (4) will have infinite number of roots, which can be expressed as follows:

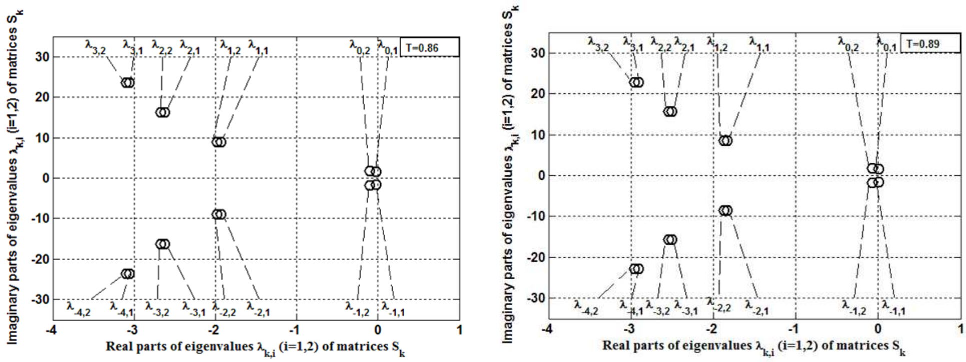
$$S_k = \frac{1}{T}W_k(-AT\mathsf{e}^{BT}) - B, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{15}$$

The value of the matrix Lambert W function $W_k(H)$ at $H = -AT\mathsf{e}^{BT}$ we obtain using the similarity transformation $H = VJV^{-1}$ [2]:

$$W_k(H) = VW_k(J)V^{-1}, \tag{16}$$

here J is the Jordan's form of the matrix H, V is the transforming matrix.

Solutions of the matrix differential equation with delayed argument (1) will be asymptotically stable if the eigenvalues of all matrices S_k, $k = 0, \pm 1, \pm 2, \ldots$ will have negative real parts.



(a) (b)

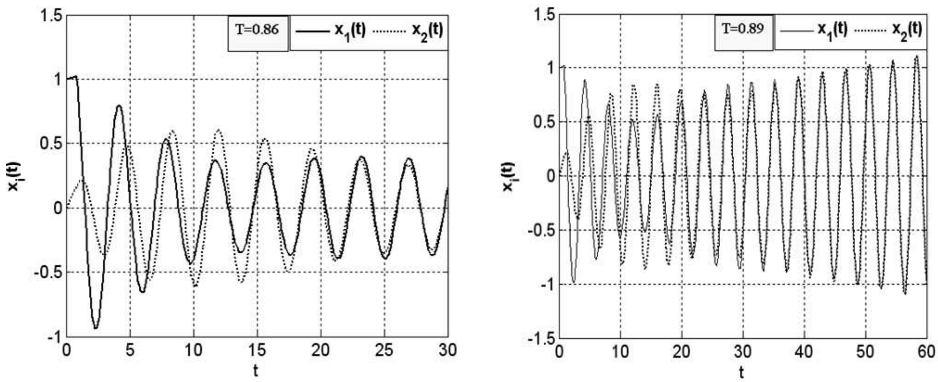
Fig. 1. Representation of the eigenvalues $\lambda_{k,i}$ of matrices S_k ($k = 0, \pm 1, \pm 2, \ldots,\ i = 1, 2$) on the complex plane: (a) real parts of all eigenvalues are negative – corresponding dynamical system is asymptotically stable; (b) real parts of eigenvalues $\lambda_{0,1}$ and $\lambda_{-1,1}$ are positive – corresponding dynamical system is not stable.

4 Results of calculations

Expressions of solutions of the matrix transcendental characteristic equation, derived in the paper, are used investigating dependence of these solutions from system's parameters. Additional information, presented in this section about transients in the corresponding systems, makes this dependence more informative. The transients are calculated numerically, using the dde programs in Matlab.

The eigenvalues $\lambda_{k,i}$, $k = 0, \pm 1, \pm 2, \ldots,\ i = 1, 2$ of the matrices S_k (the roots of the matrix characteristic equation (4), as points on the complex plane, are presented in Fig. 1(a) ($T = 0.86$) and 1b ($T = 0.89$), when $A = \begin{pmatrix} 0.004 & 0.2 \\ 0.2 & 0.004 \end{pmatrix}$, $B = \begin{pmatrix} -1.69 & 0.03 \\ 0.03 & -1.69 \end{pmatrix}$. From these figures we see, that the eigenvalues $\lambda_{0,i}$ and $\lambda_{-1,i}$, which correspond to the principal branch W_0 and the branch W_{-1} of the Lambert W function, will have the greatest influence on the asymptotic stability of solutions of matrix delay differential equation (1) (the magnitudes of the real parts of these eigenvalues are largest). The real parts of the eigenvalues remains negative (the solutions of the matrix delay differential equation remains asymptotically stable) changing T from 0 until $0,86$. This conclusion confirms the step responses of the system presented in the Fig. 2(a), (b) (the step response $x_i(t)$ ($i = \overline{1,n}$) of the system is a solution of the matrix differential equation (1), when the i-th element of the free term is the Dirac delta function $\delta(t)$ and the initial vector function $\varphi(t) = 0$, $t \in [-T, 0]$). From Fig. 2(a), (b) we see, that at $T = 0,86$ the solutions of the matrix differential equation (1) are still asymptotically stable and at $T = 0,89$ are already unstable (these results are in good agreement with the asymptotic stability analysis based on the eigenvalues of matrices S_k).

The Fig. 3 corresponds to the matrix transcendental characteristic equation (4), in which the matrices A and B are of ninth order: $A = -\kappa E$, $B = \mathrm{circ}(0, \frac{\kappa}{2}, 0, 0, 0, 0, 0, 0, 0, \frac{\kappa}{2})$ (E is an identity matrix, B is a circulant matrix).



(a) (b)

Fig. 2. Step responses of the system when the solutions of the corresponding matrix differential equation with delayed argument are: (a) asymptotically stable; (b) not stable.

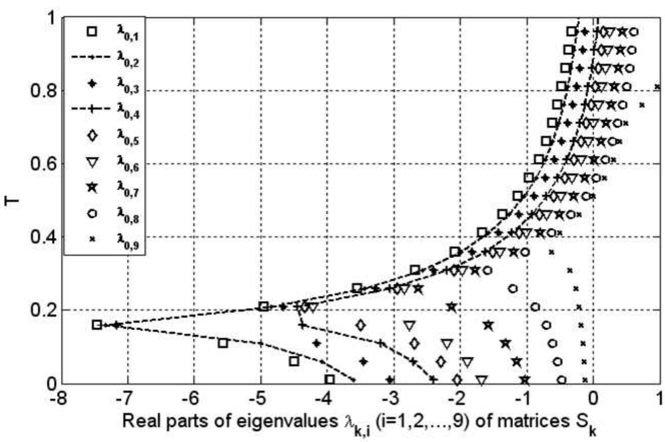


Fig. 3. Dependence of the real parts of the eigenvalues $\lambda_{k,i}$ of matrices S_k on magnitude of delay T (matrices A and B are of ninth order).

5 Conclusion

The Lambert W function method is applied to find the roots of the matrix transcendental characteristic equation corresponding to linear matrix differential equation with delayed argument. It is shown that this characteristic equation has infinitely many roots-matrices, which are values of some matrix Lambert W function in its different branches. The greatest influence to asymptotic stability of solutions of the matrix differential equation with delayed argument have the roots-matrices corresponding to the values of the Lambert W function at its principal and neighboring branch. If all eigenvalues of these roots-matrices have negative real parts, the solutions are asymptotically stable.

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REZIUMĖ

Matricinės transcendetinės charakteringosios lygties šaknų radimas taikant Lamberto W funkciją

I. Ivanovienė, J. Rimas

Tiesinę matricinę diferencialinę lygtį su vėluojančiu argumentu atitinkančios matricinės transcendentinės charakteringosios lygties šaknų radimui taikomas metodas, pagrįstas Lamberto funkcijų panaudojimu. Pateikti matricinės transcendentinės charakteringosios lygties šaknų ir jos tikrinių reikšmių skaičiavimo pavyzdžiai.

Raktiniai žodžiai: diferencialinė lygtis, vėlavimo argumentas, Lamberto W funkcija, transcendentinė charakteringoji lygtis.