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# A specialization of definitions in common knowledge logic

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**Abstract.** It is known that one of main aims of specializations of derivations in non-classical logics is the various tools which allow us to simplify the searching of termination of derivations. The traditional techniques used to ensure termination of derivations in various non-classical logics are based on loop-checking. In this paper the reflexive common knowledge logic based on modal logic $K_2$ is considered. For considered logic a sequent calculus with specialized non-logical (loop-type) axioms is presented.

**Keywords:** sequent calculus, common knowledge logic, termination.

## 1 Introduction

The common knowledge logics are import class of non-classical logics and play a significant role in several areas of computer science, artificial intelligence, game theory, economics and etc.

Common knowledge logics are based on multi-modal logics extended with common knowledge operator. In this paper, a reflexive common knowledge logic (RCKL in short) based on multi-modal logic $K_2$ and reflexive common knowledge operator [3] is considered.

Common knowledge operator satisfies some induction like axioms. In derivation this induction-like tool is realized using loop-type axioms. Determination of these loop-type axioms involves creating a new “good loop” in contrast to “bad loops” and the new loop checking along with ordinary non-induction-type loop checking. Based on history method a method of determination of “good loops” for common knowledge logic is described in [1].

In this paper for reflexive common knowledge logic (based on multi-modal logic $K_2$), a sequent calculus with specialized non-logical (loop-type) axioms is presented. The specialization is achieved using some splitting rules.

## 2 Initial calculi for considered logic

The language of considered RCKL contains:

(1) A set of propositional symbols $P_i$ $P_1, \ldots, Q_i Q_1, \ldots$;

(2) A finite set of agent constants $i,\ i_1, \ldots, i_k$ ($I,\ i_l \in \{1, \ldots, n\}$, $1 \leqslant l \leqslant k$ for simplicity we assume than $n = 2$);

(3) Multi-agent knowledge operator $K_l$, where $l$ ($l \in \{1, \ldots, n\}$) is an agent constant;

(4) Reflexive common knowledge operator $C$;

(5) Logical operators $\supset$, $\wedge$, $\vee$, $\urcorner$.

Formulas of RCKL are defined as follows: every propositional symbol is a formula; if $A$, $B$ are formulas then $A \supset B$, $A \wedge B$, $A \vee B$, $\urcorner(A)$ are formulas; if $l$ is an agent constant, $A$ is a formula, then $K_l(A)$ is a formula; if $A$ is a formula, then $C(A)$ is a formula. The formula $K_l(A)$ means "Agent $l$ knows $A$" ($l \in \{1, 2\}$). The expression $E(A)$ means "every agent knows $A$" and is used as abbreviation of formula $K_1(A) \wedge K_2(A)$. The formula $C(A)$ means "$A$ is common knowledge of all agents". The formula $C(A)$ has the same meaning as the infinite conjunction $A \wedge E(A) \wedge E(E(A)) \ldots \wedge E^K(A) \ldots)$ where $E^K(A) = E^{K-1}(E(A))$ and $E^0(A) = A$. Formal semantics of the formulas $K_l(A)$ and $C(A)$ is defined as RCKL [3]. Hilbert-type calculus HRC is obtained from Hilbert-type calculus for propositional logic by adding the following postulates:

(1) $(K_i(A \supset B) \wedge K_i(A)) \supset K_i(B)$;

(2) $(C(A \supset B) \wedge C(A)) \supset C(B)$;

(3) $E(A) = K_1(A) \wedge K_2(A)$, where $A \equiv B = (A \supset B) \wedge (B \supset A)$;

(4) $C(A) \supset (A \wedge E(C(A)))$;

(5) $(C(A \supset E(A)) \wedge A) \supset C(A)$;

(6) $\frac{A}{K^i(A)}(K_i)$;

(7) $\frac{A}{C(A)}(C)$, $i \in \{1, 2\}$.

The axiom (5) is called induction axiom. In [3] it is shown that the calculus HRC is sound and complete.

Along with formulas we consider sequents, i.e., formula expansions $A_1, \ldots, A_K \to B_1, \ldots, B_m$, where $A_1, \ldots, A_K$ and $B_1, \ldots, B_m$ is a multiset of formulas.

We consider two types of initial sequent calculi for RCKL.

1. Infinitary calculus $G_wRC$ is obtained from classical propositional calculus (with invertible logical rules) by adding the following rules:

$$\frac{\Gamma \to A}{\Pi_1, K_l\Gamma \to \Pi_2, K_l(A)}(K_l)$$

where $K_l\Gamma$ is empty or consists of formulas of the shape $K_l(B)$.

$$\frac{\Gamma \to \Delta,\ K_1(A) \wedge K_2(A)}{\Gamma \to \Delta,\ E(A)}(\to E),$$

$$\frac{\bigwedge_{l=1}^{2} K_l(A),\ \Gamma \to \Delta}{E(A),\ \Gamma \to \Delta}(E \to),$$

$$\frac{A, E(C(A)),\ \Gamma \to \Delta}{C(A),\ \Gamma \to \Delta}(C \to),$$

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to A,\ E(A); \ldots ; \Gamma \to \Delta,\ E^k(A) \ldots}{\Gamma \to \Delta,\ (A)} (\to C_w),$$

where $E^k(A) = E^{k-1}(E(A))$ and $E^0(A) = A$.

Analogously as in [2] one can prove that $G_wRC$ is sound and complete.

2. Loop-type calculus $G_LRC$ is obtained from the calculus $G_wRC$ by replacing the infinity rule $(\to \Box_w)$ by the following rule

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta,\ E(C(A))}{\Gamma \to \Delta,\ C(A)} (\to C)$$

and adding loop-type axioms as follows: a sequent $S$ is a loop-type axiom if $S$ is above a sequent $S'$ and on the same branch of a derivation, such that $S'$ can be obtained from $S$ using structural rules of weakening and contraction and there is the right premise of the rule $(\to C)$ between $S$ and $S'$.

Analogously as in [1] one can prove that $G_LRC$ is sound and complete. Therefore the calculi $G_wRC$ and $G_LRC$ are equivalent.

Let us introduce a canonical form of sequents. *A* sequent $S$ is a primary sequent, if $S$ is of the following shape $\Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2, C\Delta_2$, where for every $i$ ($i \in \{1,2\}$) $\Sigma_i$ is empty or consists of propositional symbols; $K\Gamma_i$ is empty or consists of formulas of the shape $K_l(A)$ ($l \in \{1,2\}$); $C\Delta_i$ is empty or consists of formulas of the shape $CA$

It is easy to see that bottom-up applying logical rules each sequent can be reduced to a set of primary sequents.

The primary sequent $\Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2$ is a $K$-primary one; the primary sequent $\Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, C\Delta_2$ is a $C$-primary one. The primary sequent $S$ is a non-splittable primary one if $S$ is either a $K$-primary or $C$-primary sequent. Otherwise, the primary sequent $S$ is a splittable primary one.

Let $G_L^SRC$ be a calculus obtained from the calculus $G_LRC$ by following transformations:

(1) Adding the following splitting rule

$$\frac{S_1 \text{ or } S_2}{\Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2, C\Delta_2} (Sp)$$

where the conclusion of the rule $(Sp)$ is splittable primary sequent; $\Sigma_1 \cap \Sigma_2 = \emptyset$ (i.e. the sequent $\Sigma_1 \to \Sigma_2$ is not an axiom); $S_1$ is $K$-primary sequent $\Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2$; $S_2$ is $C$-primary sequent;

(2) Replacing a loop-type axiom by specialized loop-type axiom. A specialized loop-type axiom is a loop-type axiom, which is a $C$-primary sequent;

(3) Using that the rule $(Sp)$ is admissible in the calculus $I \in \{G_wRC, G_LRC\}$ we can prove that the calculi $G_L^SRC$ and $I \in \{G_wRC, G_LRC\}$ are equivalent, therefore the calculus $G_L^SRC$ is sound and complete.

In construction of derivation it is convenient to use the following (admissible in $G_L^S RC$) rule:

$$\frac{\Gamma \to A}{\Sigma_1, E\Gamma \to E(A), \Sigma_2}(E)$$

where $\Sigma_1 \cap \Sigma_2 = \emptyset$.

*Example 1.* Let $S = P, C(P \supset E(P)| \ \to C(P), K_1(Q)$. Let's construct a derivation of $S$ in $G_L^S RC$. Since $S$ is splittable primary sequent let us backward apply to $S$ the rule $(Sp)$ and let us try to construct a derivation of $C$-primary sequent. $S_1 = P_1 C(P \supset E(P)) \to C(P)$.

$$\cfrac{\cfrac{\cfrac{\cfrac{S_1^* = P_1 C(P \supset E(P) \to C(P)}{P \to P;\ P_1 E(P),\ E(C(P \supset E(P)) \to E(C(P))}(E)}{P_1(P \supset E(P),\ E(C(P \supset E(P)) \to E(C(P))}(\supset\to)}{P_1 C(P \supset E(P)) \to P; P, C(P \supset E(P)) \to E(C(P))}(C \to)}{S_1 = P_1 C(P \supset E(P)) \to C(P)}(C \to)$$

Since $S_1 = S_1^*$, $S_1^*$ is a loop type axiom. Therefore $G_L^S RC \vdash S$.

## References

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REZIUMĖ

**Išvedimų specializacija bendrojo žinojimo logikai**
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Straipsnyje pateikiama ciklinių aksiomų specializacija refleksyviai bendro žinojimo logikai.

*Raktiniai žodžiai*: sekvencinis skaičiavimas, bendrojo žinojimo logika, ciklinės aksiomos.