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# On the left strongly prime modules and their radicals

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**Abstract.** We give the new results on the theory of the one-sided (left) strongly prime modules and their strongly prime radicals. Particularly, the conceptually new and short proof of the A.L.Rosenberg's theorem about one-sided strongly prime radical of the ring is given. Main results of the paper are: presentation of each left stongly prime ideal $\mathfrak{p}$ of a ring $R$ as $\mathfrak{p} = R \cap \mathfrak{M}$, where $\mathfrak{M}$ is a maximal left ideal in a ring of polynomials over the ring $R$; characterization of the primeless modules and characterization of the left strongly prime radical of a finitely generated module $M$ in terms of the Jacobson radicals of modules of polynomes $M\langle X_1, \dots, X_n\rangle$.

**Keywords:** strongly prime module, strongly prime ideal, primeless module, strongly prime radical, Jacobson radical.

## 1 Left strongly prime modules and ideals

All rings considered in this paper are associative with identity element which is preserved by a ring homomorphisms, all modules are unitary. $A \subset B$ means that $A$ is a proper subset of $B$.

A left non-zero module $M$ over the ring $R$ is called *strongly prime* if for any non-zero $x, y \in M$, there exits a finite set of elements $\{a_1, \dots, a_n\} \subseteq R$, $n = n(x, y)$, such that $Ann_R\{a_1x, \dots, a_nx\} \subseteq Ann_R\{y\}$, i.e., that $ra_1x = \dots = ra_nx = 0$, $r \in R$, implies $ry = 0$.

Taking $M = R$ in the definition of a strongly prime module over $R$, very important notion of a *left strongly prime ring* is obtained (see [5]). Rings that are strongly prime modules over their multiplication rings are investigated in [7].

A submodule $P$ of some module $M$ is called *strongly prime* if the quotient module $M/P$ is strongly prime $R$-module. Particularly, a left ideal $\mathfrak{p} \subset R$ is called *strongly prime* if the quotient module $R/\mathfrak{p}$ is a strongly prime $R$-module. In terms of elements, a left ideal $\mathfrak{p} \subset R$ is strongly prime if for each $u \notin \mathfrak{p}$ there exists a finite subset $\{a_1, \dots, a_n\} \subseteq R$, $n = n(u)$, such that $ra_1u, \dots, ra_nu \in \mathfrak{p}$, $r \in R$, implies $r \in \mathfrak{p}$.

Simple modules are evidently strongly prime, so maximal left ideals of a ring are strongly prime. Note, that some modules $M$ have no prime submodules. and such modules are called *primeless*. See [8] for basic properties of the primeless modules over a commutative ring.

Let us look at the quasi-injective hull $Q(M)$ of the left strongly prime module $M$. By Theorem 19.2 in [3], $Q(M) = \Lambda M \subseteq \hat{M}$, where $\hat{M}$ is the injective hull of $M$ and $\Lambda = End_R\hat{M}$.

Let $H = End_R Q(M)$, elements of which we also write from the left. So $Q(M)$ becomes a canonical left $R-H$-bimodule. Now we put the definition of the strongly prime module in the most natural context.

**Theorem 1.** *A left $R$-module $M$ is strongly prime if and only if its quasi-injective hull $Q(R)$ is the simple $R-H$-bimodule.*

*Proof.* Let $Q(M)$ be a simple $R-H$-bimodule. Take a nonzero elements $x, y \in M$. Then there exist elements $a_1, \ldots, a_n \in R$, $h_1, \ldots, h_n \in H$, such that $h_1 a_1 x + \cdots + h_n a_n x = y$. If for some $r \in R$ we have $ra_1 x = \cdots = ra_n x = 0$, then $ry = 0$ because $rh_i x = h_i rx$ for all $1 \leqslant i \leqslant n$, so $M$ is strongly prime.

Let now $M$ be strongly prime. The fact that $Q(M)$ is strongly prime $R$-module when $M$ is strongly prime is known, see [2]. Let $x, y \in Q(M)$ be a nonzero elements. Denote by $z = (a_1 x, \ldots, a_n x) \in (Q(M))^n$, where elements $a_1, \ldots, a_n \in R$ are from the definition of the strongly prime $R$-module $Q(M)$, i.e., such that $Ann_R\{a_1 x, \ldots, a_n x\} \subseteq Ann_R\{y\}$. So we can define the $R$-homomorphism $\varphi : Rz \to Ry \subseteq Q(R) \subseteq \hat{M}$ with $\varphi rz = ry$, $r \in R$. Extending $\varphi$ to the $R$-homomorphism $f : \hat{M}^n \to \hat{M}$, we obtain that $y = \varphi z = fz = \sum_k a_i h_i x$, where $h_i : Q(R) \to Q(R)$ are the restrictions the $R$-homomorphisms $f_i : \hat{M} \to \hat{M}$, $1 \leqslant i \leqslant n$, which are the components of the homomorphism $f$. This exactly means that $Q(M)$ is a simple $R-H$-bimodule. See also [10], Theorem 2.1 in Ch.13.3 for another proof of this theorem.

Let $R\langle X_H\rangle$ be a polynomial ring with the set of noncommuting indeterminates $X_h$, $h \in H$, commuting with elements of a ring $R$. We endow $Q(M)$ with the canonical $R\langle X_H\rangle$-module structure defining $X_h x = hx$ for $h \in H$ and $x \in Q(M)$. So Theorem 1 means that $Q(M)$ is a simple $R\langle X_H\rangle$-module for a strongly prime $R$-module $M$.

Let us now take a left strongly prime ideal $\mathfrak{p} \subset R$ of a ring. Taking $M = R/\mathfrak{p}$ we obtain a simple $R\langle X_H\rangle$-module $Q(R/\mathfrak{p})$, $H = End_R Q(R/\mathfrak{p})$ with an element $\overline{1}_R \in Q(R/\mathfrak{p})$. Using this element we obtain a canonical epimorphism $\psi : R\langle X_H\rangle \to Q(R/\mathfrak{p})$, sending $p \in R\langle X_H\rangle$ to the element $p\overline{1}_R \in Q(R/\mathfrak{p})$. Since $Q(R/\mathfrak{p})$ is a simple $R\langle X_H\rangle$-module, $ker\psi = \mathfrak{M}$ is a maximal left ideal in $R\langle X_H\rangle$. By the construction, $\mathfrak{M} \cap R = \mathfrak{p}$. So we obtain a very important consequence of the Theorem 1.

**Theorem 2.** *For each left strongly prime ideal $\mathfrak{p} \subset R$ there exists a maximal left ideal $\mathfrak{M} \subset R\langle X_H\rangle$, such that $\mathfrak{p} = \mathfrak{M} \cap R$.*

Thus, conceptually, the general noncommutative situation is, in some sense, similar to the commutative one since left strongly prime ideals can be obtained from a left maximal ideals in a ring of polynomials.

We can now also characterise primeless modules. Recall, that a module which has no maximal submodules is called *Jacobson-radical.* Let $\mathbf{X}$ be any set. We denote by $R\langle\mathbf{X}\rangle$ the ring of polynomials over $R$ with the set $\mathbf{X}$ of noncommuting indeterminates which commute with elements from $R$ and by $M\langle\mathbf{X}\rangle$ the polynomial module over a left $R$-module $M$. Evidently, $M\langle\mathbf{X}\rangle$ is a canonical left $R\langle\mathbf{X}\rangle$-module.

**Theorem 3.** *Module $M$ over a ring $R$ is primeless if and only if for any set $\mathbf{X}$ of indeterminates, $M\langle\mathbf{X}\rangle$ is Jacobson-radical as the $R\langle\mathbf{X}\rangle$-module.*

*Proof.* If for some set $\mathbf{X}$ the module $M\langle\mathbf{X}\rangle$ contains some maximal $R\langle\mathbf{X}\rangle$-submodule $\mathcal{N}$, then, evidently, $P = M \cap \mathcal{N}$ is strongly prime $R$-submodule in $M$, so $M$ is not primeless $R$-module. Let now $P \subset M$ be a stongly prime $R$-submodule. Then, as noted above, $Q(M/P)$ is simple $R\langle\mathbf{X}\rangle$-module, where $\mathbf{X} = X_H$. So we have the canonical $R\langle\mathbf{X}\rangle$-module epimorphism from $M\langle\mathbf{X}\rangle$ onto simple module $Q(M/P)$ and $M\langle\mathbf{X}\rangle$ is not Jacobson-radical.

## 2 Left strongly prime radical of the module

The intersection of all left strongly prime submodules of a given $R$-module $M$ is called the *left strongly prime radical* of the module $M$, which we denote by $sp_lM$. By definition, $sp_lM = M$ when $M$ does not have strongly prime submodules. First we look at the case when $M = R$. Recall, that Lewitzki radical $L(R)$ is the largest locally nilpotent ideal of the ring $R$.

**Theorem 4.** *For any ring $R$, left strongly prime radical $sp_lR$ coincides with the Lewitzki radical $L(R)$ of the ring.*

*Proof.* If some element $a \notin \mathfrak{p}$ for some left strongly prime ideal, there exist the finite set $s = \{a_1, \dots, a_n\} \subseteq Ra$, such that $ra_1, \dots, ra_n \in \mathfrak{p}$, $r \in R$, implies $r \in \mathfrak{p}$. Evidently, $s^m \nsubseteq \mathfrak{p}$ for $m \in \mathbf{N}$, so $s$ is not nilpotent subset and $a \notin L(R)$. This proves that $L(R) \subseteq sp_lR$.

Let now $a \notin L(R)$. This means, that there exists a finite subset $s = \{a_1, \dots, a_n\} \subseteq RaR$, which is not nilpotent. It's clear, that we may take the elements $a_k$ in the form $\alpha_k a \beta_k$ with $\alpha_k, \beta_k \in R$. Then the finite set $\bar{s} \subseteq Ra$, consisting of all elements of the form $\alpha_k a$ and $\beta_i \alpha_j a$ also is not nilpotent. Let $\bar{s} = \{r_1 a, \dots, r_m a\}$. It's easy to check, that the polynomial $F = (X_1 r_1 + \dots + X_m r_m)a - 1$ is not left invertible in the polynomial ring $R\langle X_1, \dots, X_m\rangle$. Thus the left ideal of the ring $R\langle X_1, \dots, X_m\rangle$, generated by the polynomial $F$, is contained in some maximal ideal $\mathfrak{M}$. Evidently $a \notin \mathfrak{M}$. By the standard fact, $\mathfrak{M} \cap R$ is the left strongly prime ideal of the ring $R$, which does not contain the given element $a$. Thus $sp_lR = L(R)$. See also for very long and complicated proof of this fact in [8].

Let now $M$ be a nonzero left finitely generated $R$-module. We denote by $M\langle X_1, \dots, X_n\rangle$ the module of polynomials over $M$ with noncommuting indeterminates $X_1, \dots, X_n$. Evidently, $M\langle X_1, \dots, X_n\rangle$ is a finitely generated module over a polynomial ring $R\langle X_1, \dots, X_n\rangle$.

It is well known, that for each nonzero finitely generated module the set of its maximal submodules is not empty. We recall, that the intersection of all maximal submodules of a given module is called the *Jacobson radical* of the module. Denote by $J_n$ the Jacobson radical of a $R\langle X_1, \dots, X_n\rangle$-module $M\langle X_1, \dots, X_n\rangle$.

**Theorem 5.** *Let $M$ be a finitely generated $R$-module. Then*

$$sp_lM = \bigcap_{n \in \mathbb{N}} (M \cap J_n).$$

*Proof.* Let $\mathcal{M} \subset M\langle X_1, \dots, X_n\rangle$ be a maximal $R\langle X_1, \dots, X_n\rangle$ submodule. Evidently $M \nsubseteq \mathcal{M}$ and $\mathcal{M} \cap M$ a proper strongly prime $R$-submodule of the module $M$. So $sp_lM \subseteq R \cap J_n$ for all $n \in \mathbb{N}$.

Let now $x_0 \notin sp_l M$ and let $M$ be generated by elements $x_1, \ldots, x_k$. This means that there exists a strongly prime submodule $P \subset M$ such that $x_0 \notin P$, so $\overline{x}_0 \neq 0$ in $\overline{M} = M/P$. As noted after the proof of the Theorem 1, the quasi-injective hull $Q(\overline{M})$ is a simple $R\langle X_H \rangle$-module, where $H = End_R Q(\overline{M})$. So we have $\overline{x}_i = p_i \overline{x}_0$ in $Q(\overline{M})$ with $p_i \in R\langle X_H \rangle$, $1 \leqslant i \leqslant k$. There is only a finite number of indeterminates from $X_H$ which occur in the polynomials $p_i$. We denote these indeterminates by $X_1, \ldots, X_n$ instead of $X_{h_1}, \ldots, X_{h_n}$ with $h_1, \ldots, h_n \in H$. Evidently, elements $x_i - p_i x_0$ belong to the kernel $\mathcal{U}$ of the canonical $R\langle X_1, \ldots, X_n \rangle$-module homomorphism $M\langle X_1, \ldots, X_n \rangle$ onto $\overline{M}\langle X_1, \ldots, X_n \rangle$. Clearly, $x_0 \notin \mathcal{U}$. By Zorn's Lemma, $\mathcal{U}$ is contained in some maximal left $R\langle X_1, \ldots, X_n \rangle$-ideal $\mathcal{M} \subset M\langle X_1, \ldots, X_n \rangle$. Also $x_0 \notin \mathcal{M}$, because $x_1, \ldots, x_k$ generate $M$. Thus we have found $R\langle X_1, \ldots, X_n \rangle$-module $MR\langle X_1, \ldots, X_n \rangle$ with $x_0 \notin J_n$.

We remark, that proved results on primeless modules and in the last theorem were not known even in the case of a commutative ring $R$.

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REZIUMĖ

**Stipriai pirminiai moduliai ir jų radikalai**

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Charakterizuoti moduliai neturintys stipriai pirminių pomodulių. Baigtinai generuotiems moduliams virš žiedo surasta stipriai pirminio radikalo išraiška per polinominių modulių Džekobsono radikalus. Gauta stipriai pirminio vienpusio idealo išraiška per maksimaliuosius polinomų žiedų idealus.


*Raktiniai žodžiai*: stipriai pirminis modulis, stipriai pirminis idealas, stipriai pirminis radikalas, Džekobsono radikalas.