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Partial cut elimination for propositional discrete linear time temporal logic

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Abstract. We consider propositional discrete linear time temporal logic with future and past operators of time. For each formula φ of this logic, we present Gentzen-type sequent calculus $G_r(\varphi)$ with a restricted cut rule. We sketch a proof of the soundness and the completeness of the sequent calculi presented. The completeness is proved via construction of a canonical model.

Keywords: sequent calculi, cut rule, temporal logic, past temporal operators, completeness.

1 Introduction

One of the simplest temporal logics which is still widely applicable is the propositional discrete linear time temporal logic *LTL*. This logic is an extension of propositional logic with two future operators: $\bigcirc$ (next, tomorrow) and U (until). The models of *LTL* are sequences of states which are infinite to the future and has the first state. Each state is a set of primitive propositions (which are true in the considered state). We use the following standard propositional connectives: $\neg$ (negation), $\vee$ (or), $\wedge$ (and), $\supset$ (implies) and propositional constants **t** (true), **f** (false). We use the standard abbreviation of $\equiv$ (equivalence).

In this paper, we consider an extension of *LTL* with past temporal operators: $\bigcirc_W$ (weak yesterday) and S (since). We denote this extension by LTL^-. We recall the semantics of the past temporal operators. For an infinite set of states $\sigma = s_0, s_1, s_2, \ldots,$ and a natural number n we define

- $\sigma, n \models \bigcirc_W \phi$ iff $n = 0$ or $\sigma, n-1 \models \phi$,
- $\sigma, n \models \phi S \psi$ iff $\exists n' \in N$: $n' \leqslant n$ and $\sigma, n' \models \psi$, and $\forall k \in N$ (if $n' < k \leqslant n$ then $\sigma, k \models \phi$).

We use the following temporal abbreviation: $\ominus\phi = \neg\bigcirc_W\neg\phi$ (strong next). We define that ϕ is *globally valid* ($\models \phi$) iff $\forall\sigma\forall n$: $\sigma, n \models \phi$.

There are several reasoning methods developed for LTL^-. The Hilbert-type axiomatic systems are presented in [7] and [4]. In the latter the tableaux-based completeness and a decision procedure are presented for the validity with respect to s_0 (anchored version).

The aim of this paper is to present the Gentzen-type sequent calculi for LTL^- which are an improvement of the temporal part of the calculi in [5]. In our paper, for each formula φ of LTL^-, we present Gentzen-type sequent calculus $G_r(\varphi)$ with a restricted cut rule. We call a cut rule *restricted* if cut formulas used in the rule are taken from some finite set (say, $\Pi(\varphi)$). We denote such a rule ($\Pi(\varphi)$-cut). Note that the temporal rules of inference ($\bigcirc$) and ($\bigcirc_W$) (from $G_r(\phi)$) are simpler than in [5]. The definition of the set of formulas $\Pi(\varphi)$ is new as well. The main results of this paper are the following: 1)we prove the soundness of $G_r(\varphi)$ using Hilbert-type calculus for LTL^-; 2)we sketch the proof of the completeness of $G_r(\varphi)$. The completeness is proved via construction of a canonical model.

2 Hilbert-type calculus for temporal logic LTL^-

We recall the Hilbert-type calculus $HLTL^-$ for LTL^- [7]. $HLTL^-$ contains the following axioms and rules of inference:

Ax All propositional tautologies and **t**,

F1 $\bigcirc(\phi \supset \psi) \supset (\bigcirc\phi \supset \bigcirc\psi)$,

P1 $\bigcirc_W (\phi \supset \psi) \supset (\bigcirc_W\phi \supset \bigcirc_W\psi)$,

F2 $(\bigcirc\neg\phi) \equiv (\neg \bigcirc \phi)$,

P2 $(\ominus\neg\phi) \supset (\neg \ominus \phi)$,

F3 $\phi U\psi \equiv \psi \vee (\phi \wedge \bigcirc(\phi U\psi))$,

P3 $\phi S\psi \equiv \psi \vee (\phi \wedge \ominus(\phi S\psi))$,

P4 $\mathbf{t} S \bigcirc_W \mathbf{f}$,

FP $\phi \supset \bigcirc \ominus \phi$,

PF $\phi \supset \bigcirc_W \bigcirc \phi$,

$$\mathbf{R1}\frac{\phi \qquad \phi \supset \psi}{\psi} \qquad \mathbf{RF1}\frac{\phi}{\bigcirc\phi} \qquad \mathbf{RF2}\frac{\phi' \supset (\neg\psi \wedge \bigcirc\phi')}{\phi' \supset \neg(\phi U\psi)}$$

$$\mathbf{RP1}\ \frac{\phi}{\bigcirc_W\phi} \qquad \mathbf{RP2}\ \frac{\phi' \supset (\neg\psi \wedge \bigcirc_W\phi')}{\phi' \supset \neg(\phi S\psi)}$$

Proposition 1 [Soundness and completeness of $HLTL^-$]. *(See [7].) For each formula φ of LTL^-, φ is provable in $HLTL^-$ iff φ is globally valid.*

3 Sequent calculi

In this section, we describe Gentzen-type sequent calculus $G_r(\varphi)$ which consists of the calculus $GLTL^-$(without cut rule) and ($\Pi(\varphi)$-cut) rule.

3.1 Preliminaries

As usually, $p, q, \ldots$ stand for primitive propositions and small Greek letters for arbitrary formulas. Further, the capital Greek letters $\Gamma, \Delta, \Sigma, \ldots$ stand for finite sets (possibly, empty) of formulas of LTL^-. $\Gamma \to \Delta$ is called a *sequent*. The semantical meaning of a sequent $\{\phi_1, \ldots, \phi_l\} \to \{\psi_1, \ldots, \psi_n\}$ is $\bigwedge_{i=1}^{i=l} \phi_i \supset \bigvee_{i=1}^{i=n} \psi_i$. For any sets Γ, Δ and a formula ϕ, the set $\Gamma \cup \{\phi\}$ is denoted by ϕ, Γ or Γ, ϕ; $\Gamma \cup \Delta$ is denoted by Γ, Δ. For a set of formulas $\Gamma = \{\phi_1, \ldots, \phi_n\}$, we use the following convenient abbreviations: $\odot\Gamma = \{\odot\phi_1, \ldots, \odot\phi_n\}$, $\odot \in \{\bigcirc, \bigcirc_W\}$.

3.2 Construction of closure sets $\widetilde{FL}(\varphi)$ and $\Pi(\varphi)$

In the introduction we have presented the notion of ($\Pi(\varphi)$-cut) rule. Now we define the finite set of formulas $\Pi(\varphi)$.

At first, for each formula φ of LTL^-, we define the set $FL(\varphi)$ which is called the *Fisher–Ladner closure* (of φ). This set is defined to be the smallest set such that: φ belongs to $FL(\varphi)$; if $\neg\psi \in FL(\varphi)$ then $\psi \in FL(\phi)$; if $\phi \vee \psi \in FL(\varphi)$ then $\phi, \psi \in FL(\varphi)$; if $\phi \wedge \psi \in FL(\varphi)$ then $\phi, \psi \in FL(\varphi)$; if $\phi \supset \psi \in FL(\varphi)$ then $\phi, \psi \in FL(\varphi)$; if $\bigcirc\psi \in FL(\varphi)$ then $\psi \in FL(\varphi)$; if $\phi U \psi \in FL(\varphi)$ then $\psi, \phi \wedge \bigcirc(\phi U \psi) \in FL(\varphi)$; if $\bigcirc_W \psi \in FL(\varphi)$ then $\psi \in FL(\varphi)$; if $\phi S \psi \in FL(\varphi)$ then $\psi, \phi \wedge \ominus(\phi S \psi), \bigcirc_W(\phi S \psi) \in FL(\varphi)$; $\mathbf{t} S \bigcirc_W \mathbf{f} \in FL(\varphi)$.

Let $|\varphi|$ denote the complexity of φ. As in [2], one can show the following:

Proposition 2. *The number of elements in* $FL(\varphi)$ *is* $c|\varphi|$, *where* c *is a constant.*

We define the finite extensions $FL'(\varphi) \subseteq \widetilde{FL}(\varphi) \subseteq \Pi(\varphi)$ of the Fisher–Ladner closure $FL(\varphi)$ as follows:

$FL'(\varphi) =_{df} FL(\varphi) \cup \{\bigcirc\neg\psi \mid \bigcirc\psi \in FL(\varphi)\} \cup \{\bigcirc_W\neg\psi \mid \bigcirc_W \psi \in FL(\varphi)\}$,

$\widetilde{FL}(\varphi) =_{df} FL'(\varphi) \cup \{\neg\psi \mid \psi \in FL'(\varphi)\}$,

$\Pi(\varphi) =_{df} \{(\wedge M_1) \vee \ldots \vee (\wedge M_k), ((\wedge M_1) \vee \ldots \vee (\wedge M_k)) \wedge \phi_1 U \phi_2, ((\wedge M_1) \vee \ldots \vee (\wedge M_k)) \wedge \phi_1 S \phi_2, \bigcirc((\wedge M_1) \vee \ldots \vee (\wedge M_k)), \bigcirc_W((\wedge M_1) \vee \ldots \vee (\wedge M_k)) \mid M_1, \ldots, M_k \subseteq \widetilde{FL}(\varphi), k \geqslant 1, \phi_1 U \phi_2, \phi_1 S \phi_2 \in \widetilde{FL}(\varphi)\}$.

Note that we get the set $\Pi(\varphi)$ by looking through the proofs of statements used to prove the Truth Theorems 2 and 3.

3.3 Gentzen-type sequent calculi with restricted cut rule

Axioms of $GLTL^-$: $\phi, \Gamma \to \Delta, \phi$; $\Gamma \to \Delta, \mathbf{t} S \bigcirc_W \mathbf{f}$; $\Gamma \to \Delta, \mathbf{t}$.

Rules of inference for propositional logical connectives (see Gentzen-type system $G4$ in [3]).

Rules of inference for temporal modalities $\bigcirc, \bigcirc_W$:

$$\frac{\Gamma \to \Delta, \bigcirc_W \Theta}{\Lambda, \bigcirc\Gamma \to \bigcirc\Delta, \Theta, \Sigma}(\bigcirc), \qquad \frac{\bigcirc\Theta, \Gamma \to \Delta}{\Lambda, \Theta, \bigcirc_W\Gamma \to \bigcirc_W\Delta, \Sigma}(\bigcirc_W).$$

In the rule $(\bigcirc)$, either ($\Theta = \emptyset$ and $\Gamma \cup \Delta \neq \emptyset$) or ($\Theta = \{\phi\}$ and $\bigcirc_W \Theta \subseteq Sub(\Gamma \cup \Delta)$). In the rule $(\bigcirc_W)$, $\Delta \neq \emptyset$ and either $\Theta = \emptyset$ or($\Theta = \{\phi\}$ and $\bigcirc\Theta \subseteq Sub(\Gamma \cup \Delta)$). Here $Sub(\Gamma \cup \Delta)$ denotes the set of subformulas of formulas from $\Gamma \cup \Delta$.

Rules of inference for temporal operators U, S:

$$\frac{\Gamma \to \Delta, \psi, \phi \wedge \bigcirc(\phi U \psi)}{\Gamma \to \Delta, \phi U \psi}(\to U), \qquad \frac{\psi, \Gamma \to \Delta; \qquad \phi \wedge \bigcirc(\phi U \psi), \Gamma \to \Delta}{\phi U \psi, \Gamma \to \Delta}(U \to),$$

$$\frac{\Gamma \to \Delta, \psi, \phi \wedge \ominus(\phi S \psi)}{\Gamma \to \Delta, \phi S \psi}(\to S), \qquad \frac{\psi, \Gamma \to \Delta; \qquad \phi \wedge \ominus(\phi S \psi), \Gamma \to \Delta}{\phi S \psi, \Gamma \to \Delta}(S \to),$$

$$\frac{\phi' \to \neg\psi \wedge \bigcirc\phi'}{\phi', \phi U \psi, \Lambda \to \Sigma}(Inv_U), \qquad \frac{\phi' \to \neg\psi \wedge \bigcirc_W \phi'}{\phi', \phi S \psi, \Lambda \to \Sigma}(Inv_S).$$

The Gentzen-type calculus $GLTL^-$ is defined. For each formula φ (of the logic LTL^-), we define the calculus $G_r(\varphi)$ to be $GLTL^-$ + ($\Pi(\varphi)$-cut) rule.

Remark 1. (Structural rules of inference) 1)one can verify that weakening rule is derivable in $G_r(\varphi)$; 2) interchange and contraction rules are implicit.

Let $\hat{G}$ be a Gentzen-type sequent calculus. We write $\hat{G} \vdash \Gamma \to \Delta$ iff there is a proof of $\Gamma \to \Delta$ in the calculus $\hat{G}$ (the notions of proof and height of proof in a Gentzen-type sequent calculus are defined as usual (see [3]).

3.4 Soundness of sequent calculi

Let $\wedge\Gamma$ ($\vee\Delta$) stand for the conjunction (resp. the disjunction) of formulas from Γ (resp. Δ). We say that a sequent calculus $\hat{G}$ is *sound* (for LTL^-) iff $\hat{G} \vdash \Gamma \to \Delta$ implies $\models (\wedge\Gamma) \supset (\vee\Delta)$ for any sequent $\Gamma \to \Delta$ of LTL^-. By the induction on the height of the given proof of $\Gamma \to \Delta$ one can verify the following:

Proposition 3. *For any sequent $\Gamma \to \Delta$ of LTL^-, if $GLTL^- + (Form\text{-}cut) \vdash \Gamma \to \Delta$ then $HLTL^- \vdash (\wedge\Gamma) \supset (\vee\Delta)$. Here $Form$ is the set of formulas of LTL^-. (Cut formulas in ($Form$-cut) rule are formulas of LTL^-.)*

Calculus $HLTL^-$ is sound for LTL^- (Proposition 1). So by Proposition 3 it follows that calculus $G_r(\varphi)$ ($= GLTL^- + (\Pi(\varphi)\text{-cut})$) is sound.

4 Completeness with restricted cut rule

In this section, we give a schema of proof of the following:

Theorem 1 [Completeness of $G_r(\varphi)$]. *For any formula φ of LTL^-, $\models \varphi$ implies $G_r(\varphi) \vdash \emptyset \to \varphi$.*

4.1 Construction of canonical model

We define a set Γ to be $\Pi(\varphi)$-*consistent* iff $G_r(\varphi) \nvdash \Gamma \to \emptyset$. The *set of states* (denoted by $W(\varphi)$) of a canonical model is defined to be maximal $\Pi(\varphi)$-consistent subsets of $\widetilde{FL}(\varphi)$. We define the binary *relation* $<$ (next) on $W(\varphi)$ as follows: $s < t$ iff $\{\psi | \bigcirc \psi \in s\} \subseteq t$ and $\{\psi | \bigcirc_W \psi \in t\} \subseteq s$.

Remark 2. 1)Note that the definition of the relation $<$ is different from the respective definition in [5]. $<$ is defined similarly as respective relation in [4]; 2)similar definition of states of a canonical model is in [1, 5, 6].

We say that an infinite sequence of states $s_0, s_1, \ldots$ is *acceptable* iff 1)for all $n \geqslant 0$, if $\phi_1 U \phi_2 \in s_n$ then there exists $l \geqslant n$ such that $\phi_2 \in s_l$; for all $n \leqslant k < l$, we have $\phi_1 \in s_k$; 2)for all $m \geqslant 0$, if $\psi_1 S \psi_2 \in s_m$ then there exists $j \leqslant m$ such that $\psi_2 \in s_j$ and, for all $j < k \leqslant m$, we have $\psi_1 \in s_k$.

A *canonical model* $\sigma^c(\varphi)$ is defined to be an infinite sequence of states $s_0, s_1, \ldots,$ (from $W(\varphi)$) such that 1)$\bigcirc_W \mathbf{f} \in s_0$; 2) for all n, $s_n < s_{n+1}$; 3) $\sigma^c(\varphi)$ is an acceptable sequence.

For a primitive proposition p we set $\sigma^c(\varphi), n \models p$ iff $p \in s_n$. For any formula $\psi \in \widetilde{FL(\varphi)}$, by the induction on $|\psi|$ we show the following:

Theorem 2 [Truth Theorem]. $\sigma^c(\varphi), n \models \psi$ *iff* $\psi \in \sigma^c(\varphi)(n)$. *(Here $\sigma^c(\varphi)(n)$ is the n-th element of $\sigma^c(\varphi)$).*

4.2 Completeness of $G_r(\varphi)$

Using properties of states from $W(\varphi)$ one can prove the following:

Theorem 3. *If $\psi \in \widetilde{FL}(\varphi)$ and ψ is a $\Pi(\varphi)$-consistent formula then the following statements hold*: 1)*there exists a state $s \in W(\varphi)$ such that $\psi \in s$*; 2) *there exists a sequence of states $\sigma = s_0, s_1, \ldots$, such that $s_n = s$ for some $n \geqslant 0$ and σ is a canonical model.*

Since $\neg\varphi \in \widetilde{FL}(\varphi)$ by Theorems 2 and 3 we have:

Corollary 1. *If $\neg\varphi$ is $\Pi(\varphi)$-consistent formula then there exists a model σ and $n \geqslant 0$ such that $\sigma, n \models \neg\varphi$ (i.e. $\not\models \varphi$).*

Theorem 1 follows by contraposition of Corollary 1.

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REZIUMĖ

Dalinis pjūvio pašalinimas teiginių diskretinei laiko logikai

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Pateikiami Gentzeno tipo sekvenciniai skaičiavimai teiginių tiesinio laiko logikai su ateities ir praeities laiko operatoriais. Šiuose skaičiavimuose pjūvio formulės priklauso baigtinei formulių aibei. Įrodomas šių skaičiavimų korektiškumas ir pilnumas.

Raktiniai žodžiai: sekvenciniai skaičiavimai, pjūvio taisyklė, laiko logika, praeities operatoriai, pilnumas.