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Infinite time ruin probability in inhomogeneous claims case

Eugenija Bieliauskienė, Jonas Šiaulys

Vilnius University, Faculty of Mathematics and Informatics
Naugarduko g. 24, LT-03225 Vilnius
E-mail: eugenija.bieliauskiene@mif.stud.vu.lt; jonas.siaulys@mif.vu.lt

Abstract. The article deals with the classical discrete-time risk model with non-identically distributed claims. The recursive formula of infinite time ruin probability is obtained, which enables to evaluate the probability to ruin with desired accuracy.

Keywords: Discrete time risk model, infinite time ruin probability, inhomogeneous claims.

Introduction

The classical discrete time risk model in Risk theory is used for investigation of insurance company's balance behavior, which main components are initial balance (or so called initial capital), premiums paid and claims paid. The main features of the model are independent, homogeneous, discrete claims and constant premiums. This model has been extensively investigated by Dickson [2], Dickson and Waters [3], De Vylder and Goovaerts [6, 7], Gerber [4], Shiu [5].

According to the classical discrete time risk model insurer's capital at each moment $n = 0, 1, 2, \dots$ is described as

$$U(n) = u + n - \sum_{i=1}^{n} Z_i.$$

Here:

- $u \in \{0\} \cup \mathbb{N}$ *is the initial insurer capital*: $u = U(0)$;
- *claim amounts* $Z_1, Z_2, Z_3, \dots$ *are assumed to be independent identically distributed* (*i.i.d.*) *non-negative integer-valued random variables* (*r.v.s*) *with local probabilities* $P(Z_1 = k) = h_k, \quad k = 0, 1, \dots,$ *and distribution function* (*d.f.*) $H(x) = P(Z_1 \leqslant x) = \sum_{k=0}^{[x]} h_k$.

One of indicators about insurance company's current risk situation is *ruin time*, which represents the first time when insurance company becomes insolvent, i.e., its balance becomes negative or null and the company is no longer able to meet its obligations:

$$T_u = \begin{cases} \inf\{n \geqslant 1 \colon U(n) \leqslant 0\}, \\ \infty, \quad \text{if } U(n) > 0 \text{ for all } n \in \mathbb{N}. \end{cases}$$

Another indicator representing the general insurance company's risk situation is a finite-time ruin probability, which represents the probability to ruin until the moment $t \in \mathbb{N}$ with initial capital $u \in \mathbb{N} \cup \{0\}$ and is defined by equality:

$$\psi(u,t) = \mathbb{P}(T_u \leqslant t).$$

The infinite time ruin probability shows the probability to ruin in any time at all:

$$\psi(u) = \mathbb{P}(T_u \leqslant \infty).$$

In case of classical discrete time risk model, the recursive formulas of finite and infinite time ruin probabilities are obtained (see, for example, [6, 3]):

$$\psi(u,1) = 1 - H(u),$$

$$\psi(u,t) = \psi(u,1) + \sum_{k=0}^{u} \psi(u+1-k, t-1) h_k, \quad t \geqslant 2,$$

$$\psi(u) = \sum_{k=0}^{u-1} (1 - H(k)) \psi(u-k) + \sum_{k=u}^{\infty} \bigl(1 - H(k)\bigr).$$

Inhomogeneous claims

The homogeneous claims' assumption in the classical risk model restricts the applications in practice, because in reality the claims are usually seasonally influenced or dependent on economic environment. Therefore the model is enhanced by allowing claims to be not necessarily identically distributed (but still independent). Then the insurer's balance is described as

$$U(n) = u + n - \sum_{i=1}^{n} Z_i,$$

and the following conditions are satisfied:

- $u = U(0) \in \{0\} \cup \mathbb{N}$;
- *claim amounts $Z_1, Z_2, Z_3, \ldots$ are independent non-negative integer-valued r.v.s. with corresponding local probabilities and distribution functions* $(j, k = 0, 1, 2, \ldots)$:

$$h_k^{(j)} = \mathbb{P}(Z_{1+j} = k), \qquad H^{(j)}(x) = \mathbb{P}(Z_{1+j} \leqslant x) = \sum_{k=0}^{[x]} h_k^{(j)}.$$

It is evident that local probabilities $h_k^{(j)}$ $(j, k = 0, 1, 2, \ldots)$ or a set of d.f.s. $H^{(j)}(x)$ $(j = 0, 1, 2, \ldots)$ describe fully the distribution of independent non-negative integer-valued r.v.s $Z_1, Z_2, Z_3, \ldots$.

We can construct the new sequence of claims $\{Z_i^{(j)}\}_{i \in \mathbb{N}}$ for every fixed $j = 0, 1, \ldots$ from the initial claim sequence by equalities $Z_i^{(j)} = Z_{i+j}$. Thus, for example,

$$\bigl\{Z_i^{(0)}\bigr\}_{i \in \mathbb{N}} = \{Z_1, Z_2, Z_3, \ldots\}, \qquad \bigl\{Z_i^{(1)}\bigr\}_{i \in \mathbb{N}} = \{Z_2, Z_3, Z_4, \ldots\}.$$

In such way described random variable sequence $Z_1^{(j)}, Z_2^{(j)}, \ldots$ $(j = 0, 1, \ldots)$ can be used for creating a shifted discrete time risk model, in which insurer's capital at each moment n is described as

$$U^{(j)}(n) = u + n - \sum_{i=1}^{n} Z_i^{(j)}.$$

In case $j = 0$ this "shifted" model coincides with the initial model.

The finite time ruin probability of such model

$$\psi^{(j)}(u,t) := \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_i^{(j)} \leqslant 0 \text{ for some } n \in \{1, 2, \ldots, t\} \right)$$

was investigated by Blaževicius, Bieliauskienė and Šiaulys [1], who obtained that $\psi^{(j)}(u,t)$ satisfy for all $j, u = 0, 1, \ldots$ the following equations:

$$\psi^{(j)}(u,1) = 1 - H^{(j)}(u), \tag{1}$$

$$\psi^{(j)}(u,t) = \psi^{(j)}(u,1) + \sum_{k=0}^{u} \psi^{(j+1)}(u+1-k, t-1) h_k^{(j)}, \quad t = 2, 3, \ldots. \tag{2}$$

However, the explicit expression of infinite time ruin probability in non-identically distributed claims model is not investigated in the literature. This note deals with the problem of deriving the infinite time ruin probability of the model with non-identically distributed claims and gives the partial formula how to calculate it.

Theorem 1. *Let us consider the discrete-time risk model with non-negative independent non-identically distributed claims as described above. Then the infinite-time ruin probabilities*

$$\psi^{(j)}(u) = \mathbb{P}\left(u + n - \sum_{i=1}^{n} Z_i^{(j)} \leqslant 0, \text{ for some } n \in \mathbb{N} \right) \tag{3}$$

for all $j = 0, 1, \ldots$ and $u = 0, 1, \ldots,$ satisfy the following equation:

$$\begin{aligned} \psi^{(j)}(u) =& \psi^{(j)}(0) + \sum_{r=1}^{u} \left(\psi^{(j)}(r) - \psi^{(j+1)}(r) H^{(j)}(u-r) \right) \\ &- \sum_{r=0}^{u-1} \left(1 - H^{(j)}(r) \right). \end{aligned} \tag{4}$$

The proof of this theorem is based on the definition of infinite time ruin probability. It is similar to a proof in a discrete-time risk model with homogeneous claims and is not presented in this article due to the lack of space.

Ballot problem

The obtained formula (4) for infinite time ruin probability enables to calculate it with a restriction, that the infinite time ruin probability with initial capital $u = 0$, i.e.,

$\psi^{(j)}(0)$ is known. Such problem is called “ballot problem” and can be solved in several ways. For example, the $\psi^{(j)}(0)$ can be evaluated by finite time ruin probability $\psi^{(j)}(0,t)$, allowing t to approach to infinity (or stopping that process when desired accuracy is reached). In addition, some restrictions on claim distributions should be assumed in order the convergence would be consistent, for example cyclical inhomogeneous claims pattern. In such case cyclical inhomogeneous claims pattern ensures that ruin probabilities repeat cyclically as well.

We observe that for independent inhomogeneous claim sequence $Z_1, Z_2, Z_3, \ldots$ with cycle k the infinite time ruin probability may be evaluated. We say, that claim sequence has cycle k, if:

$$Z_{i+nk} \stackrel{d}{=} Z_{i+(n+1)k} \quad \text{for each } i = 0, \ldots, k-1,\ n \in \mathbb{N}.$$

Example

In this section the claim sequence $Z_1, Z_2, Z_3, \ldots$ with cycle $k = 3$ is considered. In such case for all $u \geqslant 0$ and $k \in \mathbb{N}$

$$\psi^{(0)}(u) = \psi^{(3k)}(u), \qquad \psi^{(1)}(u) = \psi^{(1+3k)}(u) \quad \text{and} \quad \psi^{(2)}(u) = \psi^{(2+3k)}(u).$$

The special case when $j = 0$ and $u = 2$ is presented below in details. From formula (4) we obtain:

$$\begin{aligned}
\psi^{(2)}(2) &= \psi^{(2)}(0) + \psi^{(2)}(2) - \psi^{(0)}(2)H^{(2)}(0) \\
&\quad + \psi^{(2)}(1) - \psi^{(0)}(1)H^{(2)}(1) - \sum_{r=0}^{1}\bigl(1 - H^{(2)}(r)\bigr), \\
\psi^{(0)}(2) &= \frac{1}{H^{(2)}(0)}\left(\psi^{(2)}(0) + \psi^{(2)}(1) - \psi^{(0)}(1)H^{(2)}(1) - \sum_{r=0}^{1}\bigl(1 - H^{(2)}(r)\bigr)\right).
\end{aligned}$$

Analogously:

$$\psi^{(0)}(1) = \frac{1}{H^{(2)}(0)}\bigl(\psi^{(2)}(0) - (1 - H^{(2)}(0))\bigr)$$

and

$$\psi^{(2)}(1) = \frac{1}{H^{(1)}(0)}\bigl(\psi^{(1)}(0) - \bigl(1 - H^{(1)}(0)\bigr)\bigr).$$

Let r.v. Z_1 and Z_2 have distributions

Z_1:

k	0	1
$h_k^{(0)}$	0.5	0.5

, Z_2:

k	0	1	2	3	4
$h_k^{(1)}$	0.8	0	0	0	0.2

and Z_3 is distributed according Poisson law with $\lambda = 0.7$.

Using (1) and (2), finite time ruin probabilities are obtained, which are presented in Table 1.

It follows from this that infinite time ruin probabilities when $u = 0$ may be evaluated as: $\psi^{(0)}(0) = 0.725268$, $\psi^{(1)}(0) = 0.56957$ and $\psi^{(2)}(0) = 0.705153$

After obtaining the marginal $\psi^{(j)}(0)$ values, the necessary numerical $\psi^{(0)}(u)$, $\psi^{(1)}(u)$ and $\psi^{(2)}(u)$ values are found and presented in Table 2.

Table 1.

t	$\psi^{(0)}(0,t)$	$\psi^{(1)}(0,t)$	$\psi^{(2)}(0,t)$
1	0.5	0.2	0.503415
2	0.6	0.324644	0.503415
3	0.613657	0.324644	0.602732
4	0.613657	0.459715	0.610656
5	0.671062	0.465192	0.610656
...	...	...	...
199	0.725268	0.569578	0.705153
200	0.725268	0.569578	0.705153

Table 2.

u	$\psi^{(0)}(u)$	$\psi^{(1)}(u)$	$\psi^{(2)}(u)$
0	0.725268	0.569578	0.705153
1	0.406251	0.450536	0.461972
2	0.332169	0.361965	0.313171
3	0.229845	0.302373	0.202456
4	0.130614	0.157318	0.127967
5	0.085316	0.103909	0.081154
6	0.054489	0.066723	0.051594
7	0.034537	0.042255	0.03279
8	0.021937	0.026819	0.020828
9	0.013929	0.017055	0.013235
10	0.008863	0.010802	0.008421

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REZIUMĖ

Begalinio laiko bankroto tikimybė skirtingai pasiskirsčiusioms žaloms

E. Bieliauskienė, J. Šiaulys

Straipsnyje nagrinėjamas klasikinis diskretaus laiko rizikos modelis su skirtingai pasiskirsčiusiomis žalomis. Gaunama begalinio laiko bankroto tikimybės rekursinės formulės išraiška, kurios pagalba galima įvertinti tikimybę norimu tikslumu. Formulės veikimas iliustruojamas pavyzdžiu.

Raktiniai žodžiai: Diskretaus laiko rizikos modelis, begalinio laiko bankroto tikimybė, skirtingai pasiskirsčiusios žalos.