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The Lerch zeta-function with algebraic irrational parameter

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Abstract. In this note, we present probabilistic limit theorems on the complex plane as well as in functional spaces for the Lerch zeta-function with algebraic irrational parameter.

Keywords: Lerch zeta-function, probability measure, weak convergence.

Let $s = \sigma + it$ be a complex variable. The Lerch zeta function $L(\lambda, \alpha, s)$ with parameters $\lambda \in \mathbb{R}$ and $0 < \alpha \leqslant 1$ is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m + \alpha)^s},$$

and by analytic continuation elsewhere. For $\lambda \in \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function, while for $\lambda \notin \mathbb{Z}$, is an entire function. We consider the last case and suppose that $\lambda \in (0, 1)$.

In this note, we give a survey of the results obtained in [2,3] and [4], and correct some inaccuracies of the papers [2,3].

For $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, define

$$L(\alpha) = \{\log(m + \alpha) \colon m \in \mathbb{N}_0\}.$$

If α is algebraic irrational, then J.W.S. Cassels [1] proved that at least 51 percent of elements of the set $L(\alpha)$ are linearly independent over the field of rational numbers $\mathbb{Q}$. Denote by $I(\alpha)$ the maximal linearly independent over $\mathbb{Q}$ subset of $L(\alpha)$, and suppose that $D(\alpha) = L(\alpha) \setminus I(\alpha) \neq \varnothing$. The case $D(\alpha) = \varnothing$ is the same that of transcendental α. If $\log(m + \alpha) \in D(\alpha)$, then there exist elements $\log(m_1(m) + \alpha), \ldots, \log(m_n(m) + \alpha) \in I(\alpha)$, $n(m) < \infty$, and non-zero integers $k_0(m), k_1(m), \ldots, k_{n(m)}(m)$ such that

$$k_0(m) \log(m + \alpha) + k_1(m) \log(m_1(m) + \alpha) + \cdots + k_{n(m)}(m) \log(m_{n(m)}(m) + \alpha) = 0.$$

Hence, it follows that

$$m + \alpha = (m_1(m) + \alpha)^{-\frac{k_1(m)}{k_0(m)}} \cdots (m_{n(m)}(m) + \alpha)^{-\frac{k_{n(m)}(m)}{k_0(m)}}. \tag{1}$$

Now define two subsets $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of $\mathbb{N}_0$ by

$$\mathcal{M}(\alpha) = \{m \in \mathbb{N}_0\colon \ \log(m+\alpha) \in I(\alpha)\},$$

and

$$\mathcal{N}(\alpha) = \{m \in \mathbb{N}_0\colon \ \log(m+\alpha) \in D(\alpha)\}.$$

Moreover, let

$$\Omega = \prod_{m \in \mathcal{M}(\alpha)} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C}\colon |s| = 1\}$ for all $m \in \mathcal{M}(\alpha)$, and $\mathbb{C}$ denotes the complex plane. Then infinite-dimensional torus Ω is a compact topological Abelian group, therefore, we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S, and m_H is the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathcal{M}(\alpha)$, and if (1) holds, then, for $m \in \mathcal{N}(\alpha)$, we put

$$\omega(m) = \omega^{-\frac{k_1(m)}{k_0(m)}}\bigl(m_1(m)\bigr) \cdots \omega^{-\frac{k_{n(m)}(m)}{k_0(m)}}\bigl(m_{n(m)}(m)\bigr), \tag{2}$$

where the principal values of roots are taken. Thus, $\omega(m)$ is defined for all $m \in \mathbb{N}_0$.

We limit ourselves by a class $\mathcal{A}$ of algebraic irrational α for which the numbers

$$\frac{k_1(m)}{k_0(m)}, \ldots, \frac{k_{n(m)}(m)}{k_0(m)}$$

in (1) are integers. This requirement implies the orthogonality of the random variables $\omega(m)$, $m \in \mathbb{N}_0$. Note that in [2] and [3] the class $\mathcal{A}$ was not used, and now we correct this gap.

For $\alpha \in \mathcal{A}$ and $\sigma > \frac{1}{2}$, on $(\Omega, \mathcal{B}(\Omega), m_H)$, define the complex-valued random element $L(\lambda, \alpha, \sigma, \omega)$ by

$$L(\lambda, \alpha, \sigma, \omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m}\omega(m)}{(m+\alpha)^{\sigma}}$$

and denote its distribution by $P_L^{\mathbb{C}}$. Moreover, let $\operatorname{meas}\{A\}$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T^t(\ldots) = \frac{1}{T}\operatorname{meas}\{t \in [0, T]\colon \ldots\},$$

where in place of dots a condition satisfied by t is to be written, and the sign t in ν_T^t only indicates that the measure is taken over $t \in [0, T]$.

THEOREM 1. *Suppose that* $\lambda \in (0, 1)$, $\alpha \in \mathcal{A}$ *and* $\sigma > \frac{1}{2}$. *Then*

$$\nu_T^t\bigl(L(\lambda, \alpha, \sigma + it) \in A\bigr), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_L^{\mathbb{C}}$ *as* $T \to \infty$.

The proof of Theorem 1 remains the same as in [2].

Now let $D = \{s \in \mathbb{C}\colon \sigma > \frac{1}{2}\}$, and denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Moreover, for $\alpha \in \mathcal{A}$, on $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $L(\lambda, \alpha, s, \omega)$ by

$$L(\lambda, \alpha, s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}\omega(m)}{(m+\alpha)^s},$$

and denote its distribution by P_L^H.

THEOREM 2. *Suppose that* $\lambda \in (0, 1)$ *and* $\alpha \in \mathcal{A}$. *Then*

$$\nu_T^\tau\,(L(\lambda, \alpha, s + i\tau) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_L^H *as* $T \to \infty$.

The proof of Theorem 2 is given in [4].

Now suppose that $\alpha_1, \ldots, \alpha_r$ are distinct algebraic irrational numbers, $\lambda_j \in (0, 1)$, and $L(\lambda_j, \alpha_j, s)$, $j = 1, \ldots, r$, are the Lerch zeta-functions. Define

$$\Omega^r = \prod_{j=1}^{r} \Omega_j,$$

where

$$\Omega_j = \prod_{m \in \mathcal{M}(\alpha_j)} \gamma_m,$$

and $\gamma_m = \gamma$ for $m \in \mathcal{M}(\alpha_j)$, $j = 1, \ldots, r$. Since each torus Ω_j is a compact topological Abelian group, by the Tikhonov theorem, Ω^r is as well. Thus, we obtain a probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, where m_H^r is the probability Haar measure on $(\Omega^r, \mathcal{B}(\Omega^r))$. For brevity, denote $\underline{\omega} = (\omega_1, \ldots, \omega_r) \in \Omega^r$, where $\omega_j \in \Omega_j$, $j = 1, \ldots, r$, $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r)$. For $\alpha_1, \ldots, \alpha_r$ and $\min_{1 \leqslant j \leqslant r} \sigma_j > \frac{1}{2}$, on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the $\mathbb{C}^r$-valued random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega})$, by

$$\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) = \big(L(\lambda_1, \alpha_1, \sigma_1, \omega_1), \ldots, L(\sigma_r, \alpha_r, \sigma_r, \omega_r)\big),$$

where

$$L(\lambda_j, \alpha_j, \sigma_j, \omega_j) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_j m}\omega_j(m)}{(m+\alpha_j)^{\sigma_j}}.$$

Here $\omega_j(m)$ is the projection of $\omega_j \in \Omega_j$ to the coordinate space γ_m if $m \in \mathcal{M}(\alpha_j)$, and is the relation of type (2) otherwise. Denote by $P_{\underline{L}}$ the distribution of the random element $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega})$, i. e.,

$$P_{\underline{L}}(A) = m_H^r\left(\underline{\omega} \in \Omega^r : \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Now we are in position to state a corrected joint limit theorem on the complex plane for Lerch zeta-functions.

THEOREM 3. *Suppose that $\lambda_j \in (0,1)$, $j = 1, \ldots, r$, $\min_{1\leqslant j\leqslant r} \sigma_j > \frac{1}{2}$, and that $\alpha_1, \ldots, \alpha_r$ are distinct algebraic irrational numbers from the class $\mathcal{A}$ such that the set*

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$. Then

$$\nu_T^t\left(\underline{\omega} \in \Omega^r : \underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to the measure $P_{\underline{L}}$ as $T \to \infty$.

The proof or Theorem 3 remains the same as that of Theorem 4 from [3]. The class $\mathcal{A}$ is used only for the proof that $\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \underline{\omega})$ is a $\mathbb{C}^r$-valued random element.

Note that in the region $\sigma > 1$ the class $\mathcal{A}$ is not necessary because in this case the series

$$\sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i\lambda m}\omega(m)}{(m+\alpha)^s}$$

converges absolutely, and the orthogonality of the sequence $\{\omega(m)\colon m \in \mathbb{N}_0\}$ is not needed. So, we have the following results. Let $L_1(\lambda, \alpha, \sigma, \omega)$ be the complex-valued random element defined, for $\sigma > 1$, by the same formula as the complex-valued random variable $L(\lambda, \alpha, \sigma, \omega)$, and $P_{L_1}^{\mathbb{C}}$ is its distribution.

THEOREM 4. *Suppose that $\lambda \in (0,1)$ and $\sigma > 1$. Then*

$$\nu_T^t\left(L(\lambda, \alpha, \sigma + it) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{L_1}^{\mathbb{C}}$ as $T \to \infty$.

Let $D_1 = \{s \in \mathbb{C}\colon \sigma > 1\}$, and $H(D_1)$ is the space of analytic on D_1 functions with the topology of uniform convergence on compacta. Moreover, let $L_1(\lambda, \alpha, s, \omega)$ be the $H(D_1)$-valued random element defined by the same formula in the region D_1 as the $H(D)$-valued random element $L(\lambda, \alpha, s, \omega)$, and $P_{L_1}^H$ is the distribution of $L_1(\lambda, \alpha, s, \omega)$.

THEOREM 5. *Suppose that $\lambda \in (0,1)$. Then*

$$\nu_T^\tau(L(\lambda, \alpha, s + i\tau) \in A), \quad A \in \mathcal{B}(H(D_1)),$$

converges weakly to $P_{L_1}^H$ as $T \to \infty$.

Theorem 3 also has a similar analogue.

THEOREM 6. *Suppose that* $\lambda_j \in (0, 1)$, $j = 1, \dots, r$, $\min_{1\leqslant j\leqslant r} \sigma_j > 1$, *and that* $\alpha_1, \dots, \alpha_r$ *are distinct algebraic irrational numbers such that the set*

$$\bigcup_{j=1}^{r} I(\alpha_j)$$

is linearly independent over $\mathbb{Q}$. *Then*

$$\nu_T^t(\underline{L}(\underline{\lambda}, \underline{\alpha}, \underline{\sigma} + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to $P_{\underline{L}_1}$ *as* $T \to \infty$.

Here $P_{\underline{L}_1}$ is defined by the same formula as $P_{\underline{L}}$ with restriction $\min_{1\leqslant j\leqslant r} \sigma_j > 1$ only.

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REZIUMĖ

D. Genienė. Lercho dzeta funkcija su algebriniu iracionaliuoju parametru

Šiame straipsnyje pateikiamos tikimybinės ribinės teoremos Lercho dzeta funkcijai su algebriniu iracionaliuoju parametru kompleksinėje plokštumoje ir funkcijų erdvėse.

Raktiniai žodžiai: Lercho dzeta funkcija, tikimybinis matas, silpnas konvergavimas.