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On the uniformity of distribution of Farey fractions

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Abstract. Let $\mathcal{F}_x$ be the set of nonnegative rationals $\frac{m}{n}$ with $0 < n \leqslant x$ and $(m,n) = 1$. For some fixed interval $I \subset (0; +\infty)$, $I = (\lambda_1; \lambda_2)$ let $F(u|x, I) = \#(\mathcal{F}_x \cap (\lambda_1; \lambda_1 + u(\lambda_2 - \lambda_1))/\#(\mathcal{F}_x \cap I)$. The paper deals with the estimation of discrepancy $|F(u|x, I) - u|$, $0 \leqslant u \leqslant 1$.

Keywords: Farey fractions, uniform distribution.

Introduction

Let x be some positive integer. We denote by $\mathcal{F}_x$ the set of nonnegative rationals $\frac{m}{n}$ with $0 < n \leqslant x$ and $(m,n) = 1$. For some interval $I \subset (0; +\infty)$ let us denote $\mathcal{F}_x^I = \mathcal{F}_x \cap I$. If $I = [0; 1]$ the finite sequence of all numbers from $\mathcal{F}_x^I$, arranged in ascending order, is called the Farey sequence of order x. It is known [2,4] that some conjectures about the uniformity of distribution of Farey sequence are equivalent to the Riemann hypothesis. The following theorem is proved and discussed in [2,4].

THEOREM 1. *Let $\rho_1 < \ldots < \rho_N$ be the Farey sequence of order x, here $N = \#\mathcal{F}_x^{[0;1]}$, $\rho_N = 1$. Then the Riemann hypothesis is equivalent to the statement: the estimate*

$$\sum_{i=1}^{N} \left(\frac{i}{N} - \rho_i \right)^2 = \mathrm{O}(x^{-1+\epsilon}), \quad x \to \infty$$

holds with an arbitrary $\epsilon > 0$.

For the following development of the topic see, for example, [3].

For a moment let $I = [0; 1]$ and

$$D_x = \sup_{0 \leqslant u \leqslant 1} \left| \frac{\#(\mathcal{F}_x^I \cap [0; u])}{\#\mathcal{F}_x^I} - u \right|.$$

H. Niederreiter showed in [5] that with some absolute constants c_1 and c_2 the estimate

$$\frac{c_1}{x} \leqslant D_x \leqslant \frac{c_2}{x} \tag{1}$$

holds. More than two decades later this result was improved unexpectedly by F. Dress, who proved that indeed

$$D_x = \frac{1}{x},$$

see [1].

The purpose of this note is to establish the estimates like (1) for the discrepances related to some subsets of $\mathcal{F}_x$.

Definitions and results

Let $I = (\lambda_1; \lambda_2) \subset (0; \infty)$; the interval I may depend on x. We denote $|I| = \lambda_2 - \lambda_1$. Define the distribution function by

$$F(u|x, I) = \#\big(\mathcal{F}_x^I \cap (\lambda_1; \lambda_1 + u(\lambda_2 - \lambda_1))\big)/\#\mathcal{F}_x^I, \quad 0 \leqslant u \leqslant 1.$$

THEOREM 2. *For all $x \geqslant 1$ and I the following estimate holds*:

$$\sup_{0\leqslant u\leqslant 1} |F(u|x, I) - u| \ll \frac{1}{|I| \cdot x}.$$

The constant in $\ll$ is absolute.

As a corrollary we get immediately, that if $|I| \cdot x \to \infty$ with $x \to \infty$, then $F(u|x, I)$ converges weakly to the distribution function $F(u) = u,\ 0 \leqslant u \leqslant 1$.

THEOREM 3. *If $I = (\lambda_1; \lambda_2), \lambda_2 - \lambda_1 > 1/x$ and $\lambda_1 = a/b$ is a rational number, $(a, b) = 1$, then*

$$\sup_{0\leqslant u\leqslant 1} |F(u|x, I) - u| \geqslant \frac{1}{b|I| \cdot x}.$$

The proof of this statement is straightforward. With an arbitrary $m/n \in \mathcal{F}_x^I$

$$\frac{m}{n} - \lambda_1 \geqslant \frac{1}{bn} \geqslant \frac{1}{bx},$$

hence the interval $(\lambda_1; \lambda_1 + 1/(bx)) = (\lambda_1; \lambda_1 + u(\lambda_2 - \lambda_1))$, with $u = 1/(b|I|x)$ contains no numbers from $\mathcal{F}_x^I$. It follows then that $F(u|x, I) = 0$.

Proof of Theorem 2. Let $J = (\alpha; \beta)$ be some interval of nonnegative real numbers and

$$S(n, J) = \#\left\{\frac{m}{n}\colon (m, n) = 1,\ \frac{m}{n} \in J\right\}, \quad V(n, J) = \#\{m\colon \alpha n < m < \beta n\}.$$

Evidently, $V(n, J) = n|J| + \theta(n, J)$, here $|\theta(n, J)| \leqslant 1$. The quantity $V(n, J)$ is equal to the number of fractions $k/d \in J$, $(k, d) = 1$, $d|n$. Consequently,

$$V(n, J) = \sum_{d|n} S(d, J).$$

We have then $V(n, J) = S(n, J) * \mathbb{I}(n)$, where $*$ means the Dirichlet convolution and $\mathbb{I}(n) = 1$. Then $S(n, J) = V(n, J) * \mu(n)$, i.e.,

$$S(n, J) = \sum_{d|n} \mu\left(\frac{n}{d}\right) V(d, J).$$

Hence

$$\begin{aligned}\#\mathcal{F}_x^J &= \sum_{n \leqslant x} S(n, J) = \sum_{n \leqslant x} \sum_{d|n} \mu\left(\frac{n}{d}\right) V(d, J) \\ &= |J| \sum_{n \leqslant x} \sum_{d|n} \mu\left(\frac{n}{d}\right) d + \sum_{n \leqslant x} \sum_{d|n} \mu\left(\frac{n}{d}\right) \theta(d, J).\end{aligned}$$

Because of

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) d = \varphi(n),$$

we have

$$\#\mathcal{F}_x^J = |J| \sum_{n \leqslant x} \varphi(n) + \sum_{n \leqslant x} \sum_{d|n} \mu\left(\frac{n}{d}\right) \theta(d, J). \tag{2}$$

Let us use (2) with $J = I$ and $J = I_u = (\lambda_1; \lambda_1 + u(\lambda_2 - \lambda_1))$:

$$\begin{aligned}\#\mathcal{F}_x^I \cdot |F(u|x, I) - u| &= \left|\#\mathcal{F}_x^{I_u} - u \cdot \#\mathcal{F}_x^I\right| \\ &= \left| \sum_{n \leqslant x} \sum_{d|n} \mu\left(\frac{n}{d}\right) (\theta(d, I_u) - u\theta(d, I)) \right|.\end{aligned}$$

We denote $\theta_d = \theta(d, I_u) - u\theta(d, I)$ and rewrite the sum as

$$R = \sum_{d \leqslant x} \sum_{d|n} \mu\left(\frac{n}{d}\right) \theta_d = \sum_{d \leqslant x} \theta_d \sum_{n \leqslant x/d} \mu(n).$$

We have now

$$R \ll \sum_{d \leqslant x} \left| M\left(\frac{x}{d}\right) \right|, \quad \text{where } M(u) = \sum_{n \leqslant u} \mu(n).$$

We use in what follows the estimate $|M(u)| \ll u \exp\{-c\sqrt{\log u}\}$, where $c > 0$, $u \geqslant 2$, which follows from the law of prime number distribution. Then

$$\sum_{d \leqslant x} \left| M\left(\frac{x}{d}\right) \right| \ll x + \sum_{1 \leqslant d \leqslant x/2} \left| M\left(\frac{x}{d}\right) \right| \ll x + \sum_{1 \leqslant d \leqslant x/2} \frac{x}{d} \exp\left\{-c\sqrt{\log(x/d)}\,\right\}$$

$$\ll x + x \sum_{m=1}^{[\log(x/2)/\log 2]} \sum_{x/2^{m+1}<d\leqslant x/2^m} \frac{1}{d}\exp\{-c\sqrt{m}\} \ll x.$$

We have proved that

$$\sup_{0\leqslant u\leqslant 1} |F(u|x, I) - u| \ll \frac{x}{\#\mathcal{F}_x^I}. \tag{3}$$

Let us take $J = I$ in (2) and use the equality

$$\sum_{n\leqslant x} \varphi(n) = \frac{3}{\pi^2}x^2 + \mathrm{O}(x\log x), \quad x \to \infty.$$

Applying for the remaining term the same arguments as before for R we arrive to

$$\#\mathcal{F}_x^I = \frac{3}{\pi^2}\cdot|I|\cdot x^2\left\{1 + \mathrm{O}\left(\frac{\log x}{x} + \frac{1}{|I|\cdot x}\right)\right\}.$$

Take $c_1 > 0$ sufficiently large, such that $|I|\cdot x > c_1$ implies $\#\mathcal{F}_x^I > |I|\cdot x^2/5$. Then it follows from (3) that there exists some absolute constant c_2 such that

$$\sup_{0\leqslant u\leqslant 1} |F(u|x, I) - u| \leqslant \frac{c_2}{|I|\cdot x},$$

if $|I|\cdot x > c_1$. If we take $c_2 > c_1$, then this estimate holds trivially as $|I|\cdot x \leqslant c_1$, too. The Theorem 2 is proved.

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REZIUMĖ

V. Stakėnas. Farey trupmenų skirstinio tolygumas

Tegu $\mathcal{F}_x$ žymi neneigiamų trupmenų $\frac{m}{n}$ aibę, čia $0 < n \leqslant x$ ir $(m, n) = 1$. Intervalui $I \subset (0; +\infty)$, $I = (\lambda_1, \lambda_2)$ apibrėžkime $F(u|x, I) = \#(\mathcal{F}_x \cap (\lambda_1; \lambda_1 + u(\lambda_2 - \lambda_1))/\#(\mathcal{F}_x \cap I)$. Straipsnyje nagrinėjami nuokrypio $|F(u|x, I) - u|$, $0 \leqslant u \leqslant 1$, įverčiai.

Raktiniai žodžiai: Farey trupmenos, tolygusis skirstinys.