

Specialization of antecedent negation loop-rule for a fragment of propositional intuitionistic logic sequent calculus

Romas ALONDERIS

Institute of Mathematics and Informatics
Akademijos 4, LT-08663 Vilnius, Lithuania
e-mail: romui@ktl.mii.lt

Abstract. The paper deals with specialization of the antecedent negation loop-rule for the negative implication free fragment of the propositional intuitionistic logic.

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1. Introduction

We investigate a fragment of the propositional intuitionistic logic without negative implication. For the fragment, a loop-rule free, complete, and correct calculus $LJ_{0\setminus\supset}^{\#}$ is introduced. Idea that such a calculus could be constructed rose from Glivenko property (see [3] and [4]) which says that a propositional logic formula beginning with negation is derivable in an intuitionistic calculus iff it is derivable in classical.

We refer to [1] and [2] as to commonly known works dedicated for the loop-rule specialization problem for intuitionistic logic sequent calculi. We also mention [5] and [6] as earlier works dealing with the same problem.

The paper is organized as follows. First, we remind the common multisuccedent structural rule free Gentzen-like calculus LJ_0^* of the intuitionistic propositional logic. Using this calculus as a base, a new loop-rule free calculus $LJ_{0\setminus\supset}^{\#}$ is introduced. Then, the equivalence between LJ_0^* and $LJ_{0\setminus\supset}^{\#}$ is proved for the fragment considered.

2. Calculus LJ_0^*

The calculus LJ_0^* is a variant of the multisuccedent intuitionistic propositional Gentzen-like sequent calculus. It is defined as follows:

1. Axioms: $\Gamma, E \rightarrow E, \Delta$.

2. Rules:

$$\frac{A, B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta} (\wedge \rightarrow),$$

$$\frac{\Gamma \rightarrow A, \Delta; \Gamma \rightarrow B, \Delta}{\Gamma \rightarrow A \wedge B, \Delta} (\rightarrow \wedge),$$

$$\frac{A, \Gamma \rightarrow \Delta; B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta} (\vee \rightarrow),$$

$$\frac{\Gamma \rightarrow A, B, \Delta}{\Gamma \rightarrow A \vee B, \Delta} (\rightarrow \vee),$$

$$\frac{\neg A, \Gamma \rightarrow A, \Delta}{\neg A, \Gamma \rightarrow \Delta} (\neg \rightarrow), \quad \frac{\Gamma, A \rightarrow}{\Gamma \rightarrow \neg A, \Delta} (\rightarrow \neg),$$

$$\frac{A \supset B, \Gamma \rightarrow A, \Delta; B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta} (\supset \rightarrow), \quad \frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B, \Delta} (\rightarrow \supset).$$

Here: E denotes an atomic formula; A and B denote arbitrary formulas; Γ and Δ denote finite, possibly empty, multisets of formulas.

We introduce here some notation. We denote a derivation tree by V and the height of the derivation tree by $h(V)$. The height of a derivation tree is reckoned to be the length of the longest branch in it. The length of a branch is measured by the number of rule applications in it.

Now we present some well known properties of LJ_0^* . All LJ_0^* rules, except $(\rightarrow \neg)$ and $(\rightarrow \supset)$, are strongly invertible. I.e., if the conclusion is derivable, then so is the/each premise; moreover, there exists a derivation of any premise such that its height is less or equal than that one of the conclusion. Further. Any sequent of the shape $\Gamma, D \rightarrow D, \Delta$ is derivable (D any formula). The structural rules of weakening and contraction are strongly admissible. The rule of cut is admissible. The calculus is correct and complete for at most one formula in the succedent sequents with respect to the intuitionistic semantics.

We will freely use these properties further.

3. Calculus $LJ_{0\setminus\supset}^\#$

From now on, we consider the fragment of intuitionistic logic without negative implication.

The necessity to duplicate the main formula of $(\neg \rightarrow)$ in the premise is caused by the fact that the succedent parametric formulas (i.e., the formulas except the side ones) are dropped in the premises of $(\rightarrow \neg)$ and $(\rightarrow \supset)$. Glivenko property for our fragment can be reformulated as follows: in an LJ_0^* derivation, if the succedent of the root sequent is empty, then succedent parametric formulas can be retained in premises of $(\rightarrow \neg)$ and $(\rightarrow \supset)$ and then there is no need for the main formula to be duplicated in the premise of $(\neg \rightarrow)$. If the succedent at the root is empty, then every formula occurring in the succedent of some upper node has come from the antecedent by applying $(\neg \rightarrow)$. Here we already see a natural way how to prolong the Glivenko property for any sequents, not necessarily with the empty succedent. In the proof search, mark the succedent formulas which have come from the antecedent. Do not drop the parametric marked formulas in the premises of $(\rightarrow \neg)$ and $(\rightarrow \supset)$ and do not duplicate the main formula of $(\neg \rightarrow)$ in the premise.

The calculus $LJ_{0\setminus\supset}^\#$ is obtained from LJ_0^* by replacing its implication and negation rules. The rule $(\neg \rightarrow)$ of LJ_0^* is replaced by

$$\frac{\Gamma \rightarrow \bar{A}, \Delta}{\neg A, \Gamma \rightarrow \Delta} (LN).$$

$(\rightarrow \neg)$ and $(\rightarrow \supset)$ of LJ_0^* are replaced by

$$\frac{\Gamma, A \rightarrow \bar{\Delta}}{\Gamma \rightarrow \neg A, \bar{\Delta}, \Lambda} (RN) \quad \text{and} \quad \frac{\Gamma, A \rightarrow \overset{\alpha}{B}, \bar{\Delta}}{\Gamma \rightarrow A \overset{\alpha}{\supset} B, \bar{\Delta}, \Lambda} (RI),$$

respectively. Here α is the bar or the empty set. All formulas in $\bar{\Delta}$ are barred. No formula in Λ is barred.

When the bar is introduced in (LN) , we put it on the outermost symbol of A . If the formula is split by a rule application, the bar is put on the outermost symbols of the appropriate subformulas of A (of the side formulas) and so on. If, however, a subformula is moved into the antecedent, then the bar is dropped.

Here is a derivation of the sequent $\neg(\neg A \vee A) \rightarrow E$ in $LJ_{0 \setminus \supset}^\#$:

$$\frac{\frac{\frac{A \rightarrow \bar{A}}{\rightarrow \bar{A}, \bar{A}, E} (\rightarrow \neg)}{\rightarrow \neg A \bar{\vee} A, E} (\rightarrow \vee)}{\neg(\neg A \vee A) \rightarrow E} (\neg \rightarrow)}$$

Bars have no impact on axioms. Thus, $A \rightarrow \bar{A}$ is an axiom (we suppose here that A is atomic).

We aim $LJ_{0 \setminus \supset}^\#$ to be correct and complete for our fragment of intuitionistic sequents. However, it is not correct so far. The LJ_0^* underivable sequent $\neg(A \wedge B) \rightarrow \neg A \vee \neg B$ is derivable in $LJ_{0 \setminus \supset}^\#$:

$$\frac{\frac{\frac{A \rightarrow \bar{A}}{\rightarrow \bar{A}, \neg A, \neg B} (\rightarrow \neg)}{\rightarrow A \bar{\wedge} B, \neg A, \neg B} (\rightarrow \wedge)}{\neg(A \wedge B) \rightarrow \neg A \vee \neg B} (\neg \rightarrow), (\rightarrow \vee)}{\frac{\frac{B \rightarrow \bar{B}}{\rightarrow \bar{B}, \neg A, \neg B} (\rightarrow \neg)}{\rightarrow A \bar{\wedge} B, \neg A, \neg B} (\rightarrow \wedge)}{\rightarrow A \bar{\wedge} B, \neg A, \neg B} (\rightarrow \wedge)}{\neg(A \wedge B) \rightarrow \neg A \vee \neg B} (\rightarrow \wedge)}$$

Therefore we introduce a restriction.

RESTRICTION 1. *Neither (RI) nor (RN) with a bar-free main formula can be applied if below in the same branch $(\rightarrow \wedge)$ with a barred main formula is applied.*

4. Correctness of $LJ_{0 \setminus \supset}^\#$

LEMMA 1. *All $LJ_{0 \setminus \supset}^\#$ rules except (RN) and (RI) are strongly invertible.*

Proof. The lemma is proved by induction on the height of the conclusion derivation.

LEMMA 2. *The $LJ_{0 \setminus \supset}^\#$ rules (RN) and (RI) are strongly invertible when all the parametric formulas in the succedent of the conclusion are barred.*

Proof. Let us consider (RI):

$$\frac{\Gamma, A \rightarrow \overset{\alpha}{\bar{B}}, \bar{\Delta}}{\Gamma \rightarrow A \supset B, \bar{\Delta}} \text{ (RI)}.$$

Here α is the bar or empty set.

The base case is obvious. Let us consider the inductive case.

1) Suppose that α is the bar. Then the invertibility of (RI) is proved in the same way as invertibility of $(\rightarrow \supset)$ of the classical calculus.

2) Suppose that $\alpha = \emptyset$. Let us consider an application of (RI):

$$\frac{\Gamma, C \rightarrow \bar{D}, \bar{\Delta}}{\Gamma \rightarrow A \supset B, C \supset D, \bar{\Delta}} \text{ (RI)}.$$

We argue in the following way:

$$\frac{\frac{\Gamma, C \rightarrow \bar{D}, \bar{\Delta}}{\Gamma \rightarrow B, C \supset D, \bar{\Delta}} \text{ (RI)}}{\Gamma, A \rightarrow B, C \supset D, \bar{\Delta}} \text{ (Weakening)}.$$

It is easy to check that the rule of weakening is strongly admissible in $LJ_{0 \setminus \supset}^{\#}$.

The other cases are considered similarly as the above one or by the inductive hypothesis.

LEMMA 3. $LJ_{0 \setminus \supset}^{\#} \vdash^V \Gamma \rightarrow \bar{\Delta}, \Lambda$ implies $LJ_0^* \vdash \Gamma, \neg \Delta \rightarrow \Lambda$. Here $\neg \Delta$ is, naturally, obtained from Δ by prefixing \neg to every formula in Δ .

Proof. The lemma is proved by induction on $h(V)$.

THEOREM 1. $LJ_{0 \setminus \supset}^{\#}$ is correct with respect to the bar-free LJ_0^* derivable sequents: $LJ_{0 \setminus \supset}^{\#} \vdash S$ implies $LJ_0^* \vdash S$, where S is bar-free.

Proof. Suppose that $LJ_{0 \setminus \supset}^{\#} \vdash S$ and S is bar-free. Then, by the above lemma, $LJ_0^* \vdash S$.

5. Completeness of $LJ_{0 \setminus \supset}^{\#}$

LEMMA 4. Suppose that 1) $LJ_{0 \setminus \supset}^{\#} \vdash^V \Gamma \rightarrow \bar{\Delta}, \Lambda$ (bars do not occur in Λ), 2) ' \vee ' does not occur or occurs in Γ only in the scope of \neg , and 3) $\Lambda \neq \emptyset$. Then there is a formula $D \in \Lambda$ such that $LJ_{0 \setminus \supset}^{\#} \vdash \Gamma \rightarrow \bar{\Delta}, D$.

Proof. The lemma is proved by induction on $h(V)$. The base case is obvious. As to the inductive one, we consider only case:

$$\frac{S_1 = \Gamma \rightarrow \bar{A}, \Lambda; S_2 = \Gamma \rightarrow \bar{B}, \Lambda}{\Gamma \rightarrow A \bar{\wedge} B, \Lambda} (\rightarrow \wedge).$$

We can assume that a strategy is applied in V . Always while possible, disjunction and conjunction rules with non-barred main formulas are applied. By Lemma 1, these rules are invertible. Thus, the strategy does not narrow the class of derivable in $LJ_{0\setminus\supset}^{\#}$ -sequents.

Due to the strategy, Γ is of the shape $\Pi, \neg\Gamma'$, where Π consists of atomic formulas only. If $\Pi \rightarrow \Delta$ is an axiom, then the proof of the lemma is obvious. Otherwise, in the same way as in item 6 in the proof of Lemma 3, we show that Δ is useless for derivation and can be dropped. Now the proof of the present lemma is obvious.

LEMMA 5. *The antecedent and succedent rules of contraction are strongly admissible in $LJ_{0\setminus\supset}^{\#}$.*

Proof. The lemma is proved by induction on the ordered pair $\langle G, H \rangle$, where G is the complexity of the contraction formula and H is the height of the conclusion derivation. The rule invertibility is used, as well.

The base case is obvious. Let us consider the inductive one. We chose to consider only, more uncommon, case:

$$\frac{\Gamma, A \rightarrow \bar{B}, A\bar{\supset}B, \bar{\Delta}}{\Gamma \rightarrow A\bar{\supset}B, A\bar{\supset}B, \bar{\Delta}, \Delta} (RI).$$

By Lemma 2, $\vdash \Gamma, A, A \rightarrow \bar{B}, \bar{B}, \bar{\Delta}$. Apply twice the inductive hypothesis on G and (RI) .

LEMMA 6. *If Δ is bar-free and $LJ_{0\setminus\supset}^{\#} \vdash S = \Gamma \rightarrow \Delta$, then $LJ_{0\setminus\supset}^{\#} \vdash S' = \Gamma \rightarrow \Delta'$. Here Δ' is obtained from Δ by barring some formulas.*

Proof. The only hardship here is the Restriction 1 caused by application of $(\rightarrow \wedge)$ when the main formula is barred.

Apply $(\vee \rightarrow)$ and $(\wedge \rightarrow)$ in the bottom-up way to S and the resulting sequents while possible. Do the same for S' . We get a tree V with S at the root and a tree V' with S' at the root. The leaves of V are derivable iff the root is derivable since $(\vee \rightarrow)$ and $(\wedge \rightarrow)$ are invertible. Let us consider any leaf of V : $S_i = \Gamma_i \rightarrow \Delta$. By Lemma 4, there is $D \in \Delta$ such that $S_{i,1} = \Gamma_i \rightarrow D$. If D is not barred in S' , then we can take the derivation of $S_{i,1}$ to be the derivation of S_i and S'_i . Suppose that D is barred in S' . Note that the restriction has no impact on the derivation of $S'_{i,1} = \Gamma_i \rightarrow \bar{D}$ since it has no non-barred formulas in the succedent. Thus, $S'_{i,1}$ can be derived in the same way as $S_{i,1}$. Again, we have derivations of S_i and S'_i .

THEOREM 2. *$LJ_{0\setminus\supset}^{\#}$ is complete with respect to the non-barred LJ_0^* derivable sequents. $LJ_0^* \vdash^V S$ implies $LJ_{0\setminus\supset}^{\#} \vdash S$.*

Proof. The theorem is proved by induction on $h(V)$. The base case is obvious. The only harder case of the inductive step is the following:

$$\frac{\neg A, \Gamma \rightarrow A, \Delta}{\neg A, \Gamma \rightarrow \Delta} (\neg \rightarrow).$$

By the inductive hypothesis, $LJ_{0\setminus\supset}^{\#} \vdash \neg A, \Gamma \rightarrow A, \Delta$. By Lemma 6, $\vdash \neg A, \Gamma \rightarrow \bar{A}, \Delta$. By Lemma 1, $\vdash \Gamma \rightarrow \bar{A}, \bar{A}, \Delta$. By Lemma 5, $\vdash \Gamma \rightarrow \bar{A}, \Delta$ and

$$\frac{\Gamma \rightarrow \bar{A}, \Delta}{\neg A, \Gamma \rightarrow \Delta} (LN).$$

References

1. R. Dyckhoff. Contraction free sequent calculi for intuitionistic logic. *JSL*, 51:795–807, 1992.
2. R. Dyckhoff, S. Negri. Admissibility of structural rules for contraction-free systems of intuitionistic logic. *JSL*, 65(4):1499–1518, 2000.
3. V. Glivenko. Sur quelques points de la logique de M. Brouwer. *Bull. cl. sci. Acad. Roy. Belg.*, ser. 5, 15:183–188, 1929.
4. V. Orevkov. On Glivenko classes of sequents. *Trudy Matematicheskogo Instituta imeni V. A. Steklova*, 98:131–154, 1968.
5. N. Vorob'ev. The derivability problem in the constructive propositional calculus with strong negation. *Doklady Akademii Nauk SSSR*, 85:689–692, 1952 (in Russian).
6. N. Vorob'ev. A new algorithm for derivability in a constructive propositional calculus. *Trudy Matematicheskogo Instituta imeni V.A. Steklova*, 52:193–225, 1958. English translation: *American Math. Society Translations*, ser. 2, 94:37–71, 1970.

REZIUMĒ

R. Alonderis. Ciklinēs neigimo antecedente taisyklēs specializacija intuicionistinės propozicinēs logikos fragmento sekvenciniam skaičiamui

Straipsnyje yra pateiktas būdas kaip galima išspręsti ciklinēs neigimo antecedente taisyklēs specializacijos problemą intuicionistinės propozicinēs logikos fragmento be neigiamos implikacijos sekvenciniam skaičiamui.

Raktiniai žodžiai: sekvencinis skaičiavimas, ciklinių taisyklių specializacija.