Liet. mat. rink. LMD darbai, **50**, 2009, 241–246

Loop-free sequent calculus for modal logic *K4*

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Abstract. In the article, a loop-free calculus for modal logic *K4* is presented. The calculus is based on marks and indices method which was first used for logic *S4*.

Keywords: marks and indices, *K4*, loop-free.

1. Introduction

For such popular knowledge logics as *S4* and *K4* cut-free calculi are developed and known. These calculi provide a good way of finding derivation of derivable sequents, but are not always able to say that the sequent is not derivable. This is because derivations in the calculi can repeat themselves and form loops. A lot of effort now are put into finding an effective procedure to eliminate these loops. One of the new methods is called marks and indices. This method was first introduced in [3] for modal logic *S4* and our aim is to extend it to the logic *K4*. For other methods of effective decision procedures the reader could refer to [1] or [2].

2. Gentzen-type calculi

First we define the Gentzen-type calculus for modal logic *K4*. This definition is traditional and the loops are possible in the derivations in this calculus.

DEFINITION 1. The Gentzen-type calculus for modal logic *K4* (*GK4*) consists of axiom $\Gamma, A \to A, \Delta$, traditional propositional rules for logical operators and the modal rule of transitivity:

$$\frac{\Box\Gamma_1, \Gamma_1 \to A}{\Box\Gamma_1, \Gamma_2 \to \Delta, \Box A}\,(\Box).$$

The main cause of loops is the transitivity rule, so we must restrict the applications of this rule. To obtain the restrictions we first number all the occurrences of the indexed modality $\Box$, which can produce loops. In order to obtain termination we number all the occurrences of strongly special modality in the sequent with different indices from the set $\{\circ 1, \circ 2, \ldots\}$ and all the occurrences of simply special modality with different indices from the set $\{1, 2, \ldots\}$. These two indices are used for logic *S4* in [3] too. However we also must number all the negative occurrences of modality $\Box$ with different indices from the set $\{\ominus 1, \ominus 2, \ldots\}$. After the numbering, the sequent is called *indexed* and is denoted S_{ind}. We say that modality $\Box^i$ or $\Box^{\circ k}$ is positively indexed and modality $\Box^{\ominus l}$ is negatively indexed.

Finally to define the calculus without loops we need several more definitions.

DEFINITION 2. A decomposition of formula A is a pair $< \mathcal{A}, \mathcal{S} >$, where $\mathcal{A}$ and $\mathcal{S}$ are the sets of some subformulas of A, $\mathcal{A}$ is called an antecedental part and $\mathcal{S}$ is called a succedental part.

DEFINITION 3. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are the sets of decompositions of some formula, then the product of $\mathcal{D}_1$ and $\mathcal{D}_2$ is a set of decompositions:

$$\mathcal{D}_1 \times \mathcal{D}_2 = \big\{ \langle \mathcal{A}, \mathcal{S} \rangle : \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2, \langle \mathcal{A}_1, \mathcal{S}_1 \rangle \in \mathcal{D}_1, \langle \mathcal{A}_2, \mathcal{S}_2 \rangle \in \mathcal{D}_2 \big\}.$$

DEFINITION 4. A decomposition function $d(A, s)$, where A is some formula and $s \in \{+, -\}$, is defined in a following way:

1. $d(A, +) = \big\{ \langle \varnothing, \{A\} \rangle \big\}$, if A is a propositional variable or $A = \Box^\sigma B$, where $\sigma \in \{i, \circ k, +\}$.
2. $d(A, -) = \big\{ \langle \{A\}, \varnothing \rangle \big\}$, if A is a propositional variable or $A = \Box^\sigma B$, where $\sigma \in \{\ominus l, *\}$.
3. $d(\neg B, s) = d(B, \neg s)$, where $\neg s = \begin{cases} - & \text{if } s = + \\ + & \text{if } s = - \end{cases}$.
4. $d(B \vee C, +) = d(B, +) \times d(C, +)$.
5. $d(B \vee C, -) = d(B, -) \cup d(C, -)$.
6. $d(B \wedge C, +) = d(B, +) \cup d(C, +)$.
7. $d(B \wedge C, -) = d(B, -) \times d(C, -)$.
8. $d(B \supset C, +) = d(B, -) \times d(C, +)$.
9. $d(B \supset C, -) = d(B, +) \cup d(C, -)$.

Before defining the calculus without loops we first define a calculus, which will be used in later proofs. Here and later notation A^{i+}, $A^{\circ k+}$ and $A^{\ominus l*}$ will be used. This means that in formula A an occurrence of modality $\Box^i$ ($\Box^{\circ k}$ or $\Box^{\ominus l}$ respectively) is replaced by marked modality $\Box^+$ ($\Box^+$ or $\Box^*$). Obviously, if $\Gamma = A_1, \ldots, A_n$, then $\Gamma^\sigma = A_1^\sigma, \ldots, A_n^\sigma$, $\Gamma^\varnothing = \Gamma$ and $\Gamma^{\{\sigma_1, \ldots, \sigma_n\}} = ((\Gamma^{\sigma_1})^{\cdots})^{\sigma_n}$.

We use the same definition of primary sequent as in [3], however our calculus slightly differs from the one for *S4*.

DEFINITION 5. An indexed Gentzen-type calculus for logic *K4* ($G_1 K4$) consists of axiom $\Gamma, p \to p, \Delta$, where p is a propositional variable, traditional logical rules and rules of transitivity:

1. Antecedental transitivity rule:

$$\frac{\Box^* A_1^\tau, A_1^\tau, \ldots, \Box^* A_n^\tau, A_n^\tau, \Box^* \Gamma^\tau, \Gamma^\tau \to A^\tau}{\Box^{\ominus i_1} A_1, \ldots, \Box^{\ominus i_n} A_n, \Box^* \Gamma, \Sigma_1 \to \Sigma_2, \widetilde{\Box} \Delta, \Box^\sigma A} \ (\Box^*),$$

where $\sigma \in \{\varnothing, i, \circ k, +\}$, $\tau = \{\ominus i_1 *, \ldots, \ominus i_n *\}$ and $n >= 1$.

2. Non-indexed transitivity rule:

$$\frac{\Box^*\Gamma, \Gamma \to A}{\Box^*\Gamma, \Sigma_1 \to \Sigma_2, \Box\Delta, \Box A}\ (\Box_p).$$

3. Strong transitivity rule:

$$\frac{\Box^*\Gamma^{\sigma+}, \Gamma^{\sigma+} \to A}{\Box^*\Gamma, \Sigma_1 \to \Sigma_2, \widetilde{\Box}\Delta, \Box^\sigma A}\ (\Box_p^{\sigma+}).$$

Here $\sigma \in \{i, \circ k\}$ and if $\sigma = i$, then this rule can only be applied if the formula A does not contain any positively indexed modality and if there are no modalities of the type $\Box^{\circ k}$ in Γ.

4. Weak transitivity rule:

$$\frac{\Box^*\Gamma, \Gamma^{i+} \to A}{\Box^*\Gamma, \Sigma_1 \to \Sigma_2, \widetilde{\Box}\Delta, \Box^i A}\ (\Box_p^i).$$

This rule can only be applied if there is at least one modality of the type $\Box^{\circ k}$ in Γ.

5. Mixed transitivity rule:

$$\frac{\Box^*\Gamma^{\{\alpha_1\}\cup\beta_1}, \Gamma^{\{i+\}\cup\beta_1}, \mathcal{A}_1 \to \mathcal{S}_1\ \ldots\ \Box^*\Gamma^{\{\alpha_n\}\cup\beta_n}, \Gamma^{\{i+\}\cup\beta_n}, \mathcal{A}_n \to \mathcal{S}_n}{\Box^*\Gamma, \Sigma_1 \to \Sigma_2, \widetilde{\Box}\Delta, \Box^i A}\ (\Box_p^{i?})$$

for all $\langle \mathcal{A}_i, \mathcal{S}_i \rangle \in d(A, +)$. Moreover, for every $i = 1, \ldots, n$, if $\mathcal{S}_i$ contains at least one formula, which contains at least one positively indexed modality, then $\alpha_i = \beta_i = \varnothing$. Otherwise, $\alpha_i = i+$, and $\beta_i = \{j+ :$ A contains a positively indexed modality $\Box^j\}$. This rule can only be applied if the formula A contains at least one positively indexed modality and if there are no modalities of the type $\Box^{\circ k}$ in Γ.

6. Marked transitivity rule:

$$\frac{\Box^*\Gamma_1^{i+}, \Gamma_1^{i+} \to A}{\Box^*\Gamma, \Sigma_1 \to \Sigma_2, \widetilde{\Box}\Delta, \Box^+ A}\ (\Box_p^+).$$

DEFINITION 6. A sequent is derivable in G_1K4 if there is a derivation V and every branch of V finishes with an axiom.

Using an invertibility of logical rules we can prove the following:

LEMMA 1. *The sequent S is derivable in* GK4 *if and only if the sequent S_{ind} is derivable in* $\text{G}_1\text{K4}$.

However G_1K4 can still contain loops. The only problem now is the rule $(\Box_p^+)$. So our aim now is to eliminate this rule from the calculus.

DEFINITION 7. A Gentzen-type calculus for *K4* without loops (G_2K4) contains the same rules and axiom as calculus G_1K4, except the rule $(\Box_p^+)$.

It is easy to see that:

LEMMA 2. *If the sequent is derivable in* G_2K4*, then it is derivable in* G_1K4.

So only the reverse proof is left. To prove that, we will have to define the strategy of the derivation.

3. Termination

To define the strategy, which provides order in which rules must be chosen in the derivation, we need the definition of degree of a formula, which can be found in [3]. The definition of strategy is also based on the one found in [3].

DEFINITION 8. It is said that the derivation V in G_1K4 (or in G_2K4) is constructed using the strategy, if for any application of the rule j in V with sequent S in the conclusion the following holds:

1. If it is possible to apply some logical rule to S, then j is a logical rule.
2. Otherwise, if it is possible to apply antecedental transitivity rule to S, then j is $(\Box^*)$.
3. Otherwise, if it is possible to apply non-indexed transitivity rule to S, then j is $(\Box_p)$.
4. Otherwise, if it is possible to apply strong transitivity rule to S, then j is $(\Box_p^{i+})$ for some i.
5. Otherwise, if it is possible to apply weak (mixed) transitivity rule to S, then j is $(\Box_p^i)$ $((\Box_p^{i?}))$. If S is of the form $\Box^*\Gamma, \Sigma_1 \to \Sigma_2, \widetilde{\Box}\Delta$, then there must be no subformulas of the main formula $(\Box^i A)$ in $\widetilde{\Box}\Delta$ and:
   a) If there is a formula in $\widetilde{\Box}\Delta$, which starts with indexed modality and contains an occurrence of strongly special modality, then the formula with the largest degree from the set of $\widetilde{\Box}\Delta$ formulas that start with indexed modality and contain an occurrence of strongly special modality is chosen as the main formula.
   b) Otherwise the formula with the largest degree from the set of $\widetilde{\Box}\Delta$ formulas that start with indexed modality is chosen as the main formula.

The derivation, that is constructed using the strategy is called *ordinary derivation*.

It is possible to prove the following lemma:

LEMMA 3. *If the sequent* S_{ind} *is derivable in* G_1K4*, then it is possible to construct an ordinary derivation of* S_{ind} *in* G_1K4.

Now we have to show, that if the derivation follows the strategy, then it always terminates. This statement is true only for G_2K4 and we need several definitions to prove that. First of all, let $\mathcal{P}(A)$ denote the set of all the subformulas of formula A, that are of the form $\Box^\sigma B$, where $\sigma \in \{i, \circ k, +\}$.

DEFINITION 9. Say that we have a sequent $S = \Box^*\Gamma_1, \Box^{\ominus l}\Gamma_2, \Gamma_3 \to \Delta$ and the set Γ_3 does not contain any formula, of the form $\Box^* D$ or $\Box^{\ominus l} E$. Then the set of possible formulas of S (denoted $\mathcal{I}(S)$) is defined in a following way: for any formula $A \in \Gamma_3 \cup \Delta$, if $B \in \mathcal{P}(A)$, $B = \Box^\sigma C$, where $\sigma \in \{i, \circ k\}$, and B is not a strict subformula of any formula from $\mathcal{P}(A)$, then $B \in \mathcal{I}(S)$.

DEFINITION 10. Say that S is a sequent with the set of possible formulas $\mathcal{I}(S)$. Then formula $A \in \mathcal{I}(S)$ is preferable if any formula $B \in \mathcal{I}(S)$ is not a strict subformula of A.

The termination of the derivation in *G_2K4* can be shown by providing a tuple, which always decreases in the derivation. The tuple is similar to the one in [3].

DEFINITION 11. A complexity of a sequent S is an ordered tuple $C(S) = \langle k(S), r(S), m(S), p(S), l(S)\rangle$, where:

- $k(S)$ is the number of formulas of the type $\Box A$ in the succedent of S;
- $r(S)$ is the number of different indices of type $\ominus l$ in S;
- $m(S)$ is the number of different indices of type i and $\circ k$ in S;
- $p(S)$ is the number of different subformulas of the shape $\Box^i A$ and $\Box^{\circ k} A$ of the formulas from the set of preferable formulas of the sequent S;
- $l(S)$ is the sum of the lengths of formulas of sequent S, which is defined in a traditional way.

It can be shown that after the application of the rule $(\Box_p)$ the value of the function k decreases. After the application of $(\Box^*)$ the value of k stays the same, but the value of r decreases. After the application of $(\Box_p^{i+})$ the values of k and r do not increase and the value of m decreases. After the application of $(\Box_p^i)$ or $(\Box_p^{i?})$ the values of k, r and m do not increase and the value of p decreases and after the application of logical rule the values of k, r, m and p do not increase but the value of l decreases. So to sum up, we can prove the following:

THEOREM 1 (termination of derivation). *Let P be an ordinary derivation in* G_2K4. *Then for any application of rule j in the proof with the conclusion S and premises $S_1, \ldots, S_n$ for all $i = 1, \ldots, n$ $C(S) > C(S_i)$.*

From this theorem and from the definition of calculus *G_2K4* the following can be proved:

COROLLARY 1. *The ordinary derivation of sequent in* G_2K4 *always terminates. More precisely, all the branches of the derivation always finishes either with an axiom or with a sequent of the form $\Box^*\Gamma, \Sigma_1 \to \Sigma_2, \Box^+\Delta$, which is called a* final sequent. *Here $\Sigma_1 \cap \Sigma_2 = \varnothing$ and both Σ_1 and Σ_2 contain only propositional variables.*

So the calculus *G_2K4* does not contain loops, however backtracking is still needed. Having in mind the invertibility of the logical rules, the search of the derivation could contain the following steps:

1. The logical rules are applied as long as they can be applied. We do not need to return to these applications, because all the logical rules are invertible.
2. If the sequent is not final, the transitivity rule is chosen according to the strategy.
3. If the sequent is final, we must go back to the last application of the transitivity rule and consider other main formulas, if they are possible according to the strategy.

4. Completeness

To show the completeness of the calculus G_2K4, we need to show that the rule ($\Box_p^+$) is not needed for the calculus G_1K4. This can be demonstrated by proving the following:

LEMMA 4. *If the sequent S_{ind} is derivable in* $\mathrm{G_1K4}$*, then any ordinary derivation V of S does not contain final sequents.*

Having in mind Lemma 4 and the fact that according to Definition 8 in ordinary derivations the rule ($\Box_p^+$) can only be applied to the final sequents, we can show that:

COROLLARY 2. *If the sequent S_{ind} is derivable in* $\mathrm{G_1K4}$*, then any ordinary derivation V of S does not contain applications of rule ($\Box_p^+$).*

The completeness of the calculus G_2K4 follows easily:

LEMMA 5. *If the sequent S_{ind} is derivable in* $\mathrm{G_1K4}$*, then it is derivable in* $\mathrm{G_2K4}$*.*

Finally, using Lemmas 1 and 5 we get the proof of the final theorem.

THEOREM 2. *The sequent S is derivable in* GK4 *if and only if the sequent S_{ind} is derivable in* $\mathrm{G_2K4}$*.*

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REZIUMĖ

J. Andrikonis. Beciklis sekvencinis skaičiavimas modalumo logikai K4

Straipsnyje pateikiamas modalumo logikos *K4* skaičiavimas be ciklų. Šis skaičiavimas yra paremtas žymių ir indeksų metodu, kuris pirmą kartą buvo panaudotas logikai *S4*.

Raktiniai žodžiai: žymių ir indeksų metodas, K4, skaičiavimas be ciklų.