Liet. mat. rink. LMD darbai, **50**, 2009, 253–257

About the equivalent replaceability of the double induction axiom

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Abstract. In this paper the first order predicate calculus with the axioms of additive arithmetic is investigated. The conditions of the equivalent replaceability of a double induction axiom is presented.

Keywords: sequent calculus, additive arithmetic, double induction axiom.

1. Introduction

In the systems without the restricted difference the axiom of double induction (ADI) is definitely stronger than the usual axiom of induction. J.R. Shoenfield [1] was investigating (by using the models theory) the replaceability of the usual induction rule (RI) with the open induction formula in additive arithmetic, containing the axioms B1–B6 and showed that the axioms B5 and B6 are not provable by RI, but are provable by using the rule of double induction (RDI). J.C. Sheperdson [2] has risen also a question – how strong RDI is.

In this paper we will finish the investigation (which has begun in [3,4] and [5]) of equivalent replaceability of a double induction axiom in the additive arithmetic.

2. Description of basic calculi

Let Z_0 be the sequential variant of the first order predicate calculus with the signature {= (equality), 0 (zero), $'$ (successor), P (predecessor), + (plus)} and usual rules for the logical symbols (see, e.g., [3]); structural rules, cut rule

$$\frac{\Gamma \to Z, \mathcal{F};\ \mathcal{F}, \Delta \to \Lambda}{\Gamma, \Delta \to Z, \Lambda},$$

substitution rules

$$\frac{\Gamma_d^\alpha,\ c=d,\ \Delta_d^\alpha \to Z_d^\alpha}{\Gamma_c^\alpha,\ c=d,\ \Delta_c^\alpha \to Z_c^\alpha}\ (S_1) \qquad \frac{\Gamma_d^\alpha,\ d=c,\ \Delta_d^\alpha \to Z_d^\alpha}{\Gamma_c^\alpha,\ d=c,\ \Delta_c^\alpha \to Z_c^\alpha}\ (S_2)$$

(where $\Gamma, \Delta, Z, \Lambda$ are finite (probably, empty) sequences of the formulae; the expression Γ_β^α denotes substitution β for every occurrence of α in every formula of Γ; $\mathcal{F}$ is formula, t is an arbitrary term), axioms

$$\text{P1. } \Gamma, \mathcal{F}, \Delta \to Z, \mathcal{F}, \Lambda, \qquad \text{P2. } \Gamma \to Z, t=t, \Lambda,$$

A1. $\rightarrow t' \neq 0$,
A2. $\rightarrow P0 = 0$,
A3. $\rightarrow Pt' = t$,
A4. $\rightarrow t + 0 = t$,
A5. $\rightarrow t + s' = (t + s)'$
and the axiom of double induction (ADI):

$$\forall x \mathcal{A}(x, 0) \& \forall y \mathcal{A}(0, y) \& \forall xy[\mathcal{A}(x, y) \supset \mathcal{A}(x', y')] \rightarrow \forall xy \mathcal{A}(x, y)$$

for all $\exists$-reduced formulae $\mathcal{A}(x, y)$.

Let $\tilde{Z}$ be the system, obtained form Z_0 by replacement of the ADI by the following axioms:
B1. $\rightarrow t \neq 0 \supset (Pt)' = t$,
B2. $\rightarrow t + s = s + t$,
B3. $\rightarrow (t + s) + r = t + (s + r)$,
B4. $\rightarrow t + s = t + r \supset s = r$,
B5. $\rightarrow nt = ns \supset t = s$, $n = 2, 3, \ldots$,
B6. $\rightarrow mt + n \neq ms$, $0 < n < m$,
B7. $\rightarrow t \neq s \supset \exists r(t + r' = s) \vee \exists w(t = s + w')$.

We shall call the formulas of the form $mx + q = t$, $my + q = t$, $mx + ny + q = t$, $mx + q = ny + t$, where x, y are free variables; $m \neq 0$, $n \neq 0$ are the natural numbers; q, t be a terms, that do not contain the variables x and y, *substantial elementary* formulas and the formulas of the form $x = y \vee \exists c(x + c' = y) \vee \exists c(x = c' + y)$ – $\exists$-*elementary* formulas of the calculus Z_0 (also and of the calculus $\tilde{Z}$).

The formula $\mathcal{A}(x, y)$ of the calculus $\tilde{Z}$ we shall call $\exists$-*reduced*, if there is of the form

$$\bigvee_{i \in \mathcal{U}} B_i(x, y) = \bigvee_{i \in \mathcal{U}} \Big(\underset{j \in R_{1i}}{\&} \mathcal{E}_{ij}(x, y) \underset{j \in R_{2i}}{\&} \neg\, \mathcal{E}_{ij}(x, y) \underset{j \in R_{3i}}{\&} \tilde{\mathcal{E}}_{ij}(x, y) \,\&\, D_i \Big), \qquad (1)$$

where $\mathcal{E}_{ij}(x, y)$ are substantial elementary formulas; $\tilde{\mathcal{E}}_{ij}(x, y)$ are $\exists$- elementary formulas or the negations of theirs; D_i is a formula that does not contain variables x and y; $\mathcal{U}$, R_{1i}, R_{2i}, R_{3i} $(i \in \mathcal{U})$ are finite subsets of the set of natural numbers, and the definition of the set $\mathcal{U}$ is detailed in such way:
1) $\mathcal{U} = \bigcup_{k=1}^{4} \mathcal{U}_k$, $\bigcap_{k=1}^{4} \mathcal{U}_k = \varnothing$,
2) when $i \in \mathcal{U}_1$, then $R_{1i} \neq \varnothing$,
3) when $i \in \mathcal{U}_2$, then $R_{1i} = \varnothing$ and $R_{2i} \neq \varnothing$,
4) when $i \in \mathcal{U}_3$, then $R_{1i} = R_{2i} = \varnothing$, $R_{3i} \neq \varnothing$,
5) when $i \in \mathcal{U}_4$, then $R_{1i} = R_{2i} = R_{3i} = \varnothing$.

3. The basic theorems

THEOREM 1. *Let $\mathcal{A}(x, y)$ be an $\exists$-reduced formula of the calculus $\tilde{Z}$, then*[1]

$$\vdash_{\tilde{Z}} ADI.$$

Really, if the induction formula $\mathcal{A}(x, y)$ is open, then we shall in usual way (see, e.g., [3]) reconstruct it into the disjunctive normal form

$$\bigvee_{i\in\mathcal{U}} B_i(x, y) = \bigvee_{i\in\mathcal{U}} \Big(\mathop{\&}_{j\in R_{1i}} \mathcal{E}_{ij}(x, y) \mathop{\&}_{j\in R_{2i}} \neg\, \mathcal{E}_{ij}(x, y) \,\&\, D_i\Big),$$

where $\mathcal{E}_{ij}(x, y)$ are substantial elementary formulas; $\mathcal{U}$, R_{1i}, R_{2i} $(i \in \mathcal{U})$ are finite subsets of the set of natural numbers; D_i is a formula that does not contain variables x and y.

If $\mathcal{A}(x, y)$ is an $\exists$-reduced formula, its d.n.f. has a shape (1).

Let η be the number of the elements of the set $\mathcal{U}$; μ_i be the number of the conjuncts in the disjunct $B_i(x, y)$ and we shall mark

$$\sigma = \eta + \max_{1\leqslant i\leqslant \eta} \mu_i.$$

The consequences of the sequent

$$\forall x \mathcal{A}(x, 0) \& \forall y \mathcal{A}(0, y) \& \forall xy[\mathcal{A}(x, y) \supset \mathcal{A}(x', y')] \to \forall xy \mathcal{A}(x, y)$$

are the sequents

$$\mathop{\&}_{\xi=0}^{\mu_i} (B_i(\nu_\xi + \varepsilon,\ \nu_\xi) \& B_i(\tau_\xi + \omega,\ \tau_\xi)),\ \Gamma \to \Delta, \tag{2}$$

where $\Gamma \rightleftharpoons \forall x \mathcal{A}(x, 0) \& \forall y \mathcal{A}(0, y) \& \forall xy[\mathcal{A}(x, y) \supset \mathcal{A}(x', y')]$; $\Delta \rightleftharpoons \mathcal{A}(a, b)$; a, b be parameters that do not occur in Γ; $\nu, \tau, \varepsilon, \omega, \xi \in N$; $\omega \neq \varepsilon$; $\nu_l \neq \nu_k, \tau_l \neq \tau_k$, if $l \neq k$; $l, k \in N$.

For the sequent (2) are possible the following cases: $i \in \mathcal{U}_1$ or $i \in \mathcal{U}_2$, then proof of the sequent (2) is constructed analogously, as in [4] (see Lemma 2 and Theorem 2). If $i \in \mathcal{U}_3$, then $R_{1i} = R_{2i} = \emptyset$ and

$$B_i(x, y) = \mathop{\&}_{j\in R_{3i}} \tilde{\mathcal{E}}_{ij}(x, y) \,\&\, D_i,$$

where $\tilde{\mathcal{E}}_{ij}(x, y)$ are an $\exists$-elementary formulas or the negations of theirs; D_i is a formula that does not contain variables x and y. Let

$$R_{3i} = \bigcup_{k=1}^{4} R_{3i_k}, \qquad \bigcap_{k=1}^{4} R_{3i_k} = \emptyset.$$

[1] The expression $\vdash_Z Q$ will denote, that the object Q is deducible in the calculus Z.

If $j \in R_{3i_1}$ (analogously, R_{3i_2}; R_{3i_3}; R_{3i_4}) then $\tilde{\mathcal{E}}_{ij}(x, y) = \exists c_{ij}(x + c'_{ij} = y)(\exists c_{ij}(x = c'_{ij} + y)$; $\neg\exists c_{ij}(x + c'_{ij} = y)$; $\neg\exists c_{ij}(x = c'_{ij} + y))$.

Let $R_{3i_1} \neq \emptyset$, then we can get from the sequent (2) the sequent $\to (\varepsilon + h)' \neq 0$, i.e., the axiom A1. Analogously, if $R_{3i_1} = \emptyset$, $R_{3i_2} \neq \emptyset$. In the cases $R_{3i_1} = R_{3i_2} = \emptyset$, $R_{3i_3} \neq$ or $R_{3i_4} \neq \emptyset$, the consequences of the sequent (2) are

$$a + q' = b, \Gamma \to \Delta \quad \text{or} \quad a = q' + b, \Gamma \to \Delta$$

(where q is a term that not occur in Γ). These sequents can be proved in analogical way like in Theorem 1 of [3].

THEOREM 2. *The calculus Z_0 and $\tilde{Z}$ are equivalent*:

$$Z_0 \Leftrightarrow \tilde{Z}.$$

Proof. Part I. The axiom ADI with $\exists$-reduced induction formula $\mathcal{A}(x, y)$ is provable in the calculus $\tilde{Z}$ by Theorem 1.

Part II. We must show that the all axioms of the calculus $\tilde{Z}$ are provable in the calculus Z_0. The axioms B1–B4 are provable by the usual induction axiom (see, e.g., [1,2]), so they all the more are provable by the ADI. The axioms B5, B6 can not be proved by the usual induction axiom, we can prove them by ADI (see [4]). Let

$$\mathcal{A}(t, s) \rightleftharpoons t \neq s \supset \exists r(t + r' = s) \vee \exists \omega(t = \omega' + s),$$

then basis for the axiom B7 is the sequent

$$\to \mathcal{A}(t, 0), \quad \text{i.e.,} \ \to t \neq 0 \supset (\exists r(t + r' = 0) \vee \exists \omega(t = \omega')),$$

and from that follow $\to t \neq 0 \supset t = (Pt)'$, i.e., the axiom B1. The provability of the sequent $\to \mathcal{A}(0, s)$ is showed analogously.

Induction step. The proof of the sequent

$$\to (t \neq s \supset \Gamma) \supset (t' \neq s' \supset \Delta),$$

(where $\Gamma \rightleftharpoons \exists r(t + r' = s) \vee \exists w(t = s + w')$; $\Delta \rightleftharpoons \exists r(t' + r' = s') \vee \exists w(t' = s' + w')$) consisting of the branches:

$$\to t' \neq s' \supset t \neq s, \Delta; \quad t' \neq s', \exists r(t + r' = s) \to \exists r(t' + r' = s');$$

$$t' \neq s', \ \exists \omega(t = \omega' + s) \to \exists \omega(t' = \omega' + s').$$

The all of theirs can be reduced to the sequents

$$\to t' = t'; \quad \to s' = s' \quad \text{and} \quad \to t' = t'$$

accordingly, i.e., to the axiom P2.

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REZIUMĖ

L. Maliaukienė. Apie ekvivalentų dvigubos indukcijos aksiomos pakeičiamumą

Nagrinėjamas ekvivalentus dvigubos indukcijos aksiomos (ADI) pakeičiamumas pirmos eilės predikatų skaičiavime su lygybe ir papildomais simboliais $\{0, ', P, +\}$. Pateikiama baigtinė, neturinti ADI, aksiomų sistema, ekvivalenti pradiniam skaičiavimui.

Raktiniai žodžiai: sekvencinis pirmos eilės predikatų skaičiavimas, adicinė aritmetika, dvigubos indukcijos aksioma.