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Elimination of loop-check for logic of idealized knowledge

Aida PLIUŠKEVIČIENĖ
Institute of Mathematics and Informatics
Akademijos 4, LT-08663 Vilnius, Lithuania
e-mail: aida@ktl.mii.lt

Abstract. In the paper loop-check-free sequent calculus for logic of idealized knowledge is presented. To obtain termination of derivation indices and marks are used instead of history.

Keywords: modal logics, sequent calculus, termination of derivation, loop-check, indexation.

1. Introduction

Traditional techniques used for test termination of backward proof search in modal (e.g., knowledge-based) sequent (and tableau) calculi are based on *loop-check*. Namely, before applying any rule it is checked if this rule was already applied to "essentially the same" sequent; if this is the case we block the application of the rule. In [2,4] efficient loop-check for modal logics KT, $K4$, $S4$, tense logic K_t, and a fragment of intuitionistic logic was presented using sequents extended by notion of history. In [5] a contraction-free (i.e., loop-check-free) calculus for mono-modal logic $S5$ is presented. In this paper instead of loop-check and history the method of indexation is used. With a view to construct a cut-free calculus for $S5$ Kanger [3] has introduced the index method. Later on Fitting [1] has developed the prefixed tableau method for various modal logics. Here multi-modal logic $S5$ is considered. This modal logic is used in artificial intelligence and computer science and is considered as the logic of idealized knowledge. The aim of this paper is to get a specialization of derivations for the multi-modal logic $S5$ that allows us to present loop-check-free decision procedure. To this aim we have used indexing as well and construct invertible loop-check-free sequent calculus. The result presented in the paper is an extension and revision of the results obtained in [6]. In the paper the multi-modal logic $S5$ when $n = 2$ (in notation $S5_2$) is considered.

2. Initial sequent calculus for $S5_2$

To get cut-free sequent calculus for mono-modal logic $S5$ Kanger proposed to use indexed propositional symbols along with usual propositional ones [3]. The indices in Kanger-style sequent calculus for $S5$ are arbitrary natural numbers. In Kanger-style sequent calculus $KS5_2$ for logic $S5_2$ a list (possibly empty) consisting of the ordered pairs $\langle k, l \rangle$ (where $k \in \{1, 2\}$ and l is either zero or an arbitrary natural number) is used as index. Let P be a propositional symbol, then P^γ is an indexed propositional symbol. Let γ be any index, then indexation procedure of formula in $S5_2$ is as follows:

1. $(P^\gamma)^{\langle i,r\rangle} = P^{\gamma,\langle i,r\rangle}$, where $i \in \{1,2\}$;
2. $(A \odot B)^\gamma = A^\gamma \odot B^\gamma$, where $\odot \in \{\supset, \wedge, \vee\}$.
3. $(\neg A)^\gamma = \neg(A)^\gamma$;
4. $(\Box_j A)^\gamma = \Box_j (A)^\gamma$ $(j \in \{1,2\})$.

In addition, for arbitrary indices γ and γ_1 it is assumed that $P^{\gamma,\langle i,0\rangle,\gamma_1} = P^{\gamma,\gamma_1}$, and $P^{\gamma,\langle i,l\rangle,\langle i,r\rangle\gamma_1} = P^{\gamma,\langle i,r\rangle,\gamma_1}$, where $i \in \{1,2\}$.

A sequent is a formal expression $\Gamma \to \Delta$, where Γ, Δ are multisets of formulas.

Let $KS5_2$ be a calculus obtained from invertible Kanger-style logical calculus [3] adding the following modal rules:

$$\frac{\Gamma \to \Delta, A^{\langle i,r\rangle}}{\Gamma \to \Delta, \Box_i A}\ (\to \Box_i) \qquad \frac{A^{\langle i,\alpha\rangle}, \Box_i A, \Gamma \to \Delta}{\Box_i A, \Gamma \to \Delta}\ (\Box_i \to),$$

where $i \in \{1,2\}$; in the rule $(\to \Box_i)$ a natural number $r \in \{1,2,\ldots\}$ is such that any index in the conclusion does not contain a pair $\langle i,r\rangle$; in the rule $(\Box_i \to)$ value of metavariable α is either zero or natural number defined as follows: if any index in the conclusion does not contain a pair of the shape $\langle i,b\rangle$ where b is an arbitrary natural number, then α is zero, otherwise value of metavariable α is natural number $l \in \{1,2,\ldots\}$ such that $\langle i,l\rangle$ enters in some index of the conclusion.

As in [6] the following theorems can be proved.

THEOREM 1. *The calculus $KS5_2$ is a conservative extension of a traditional Hilbert-style calculus $HS5_2$, i.e., the calculus $KS5_2$ is sound and complete.*

THEOREM 2. *The structural rules of weakening, contraction, and cut are admissible in $KS5_2$.*

From the admissibility of the weakening rules it follows the invertibility of the rule $(\Box_i \to)$.

A derivation in a calculus I is called an atomic one if the main formula of an axiom is a propositional symbol. It is obvious that backward applying rules of $KS5_2$ each derivation in $KS5_2$ can be reduced to an atomic one with the same end-sequent.

By induction on the height of derivation V denoted by $h(V)$ we can prove

LEMMA 1. *Let S (S_1) be a conclusion (premise, correspondingly) of a logical rule or the rule $(\to \Box_i)$. Let $KS5_2 \vdash^V S$ where V is an atomic derivation of S in $KS5_2$ and $h(V)$ is a height of this derivation. Then $KS5_2 \vdash^{V^*} S_1$ and $h(V^*) < h(V)$.*

A backward proof search in the calculus $KS5_2$ is not terminative, in general. Indeed, let S be a sequent $\Box_i(P \vee Q) \to P$. Then the backward proof search contains an infinitive branch because we repeatedly get almost the same sequents $S_m = \overbrace{Q,\ldots,Q}^{m\ times}, \Box_i(P \vee Q) \to P, m \in \{1,2,\ldots\}$.

To prune the infinite branch the method of loop-check [1] is used. Since the sequents S_1 and S_2 are almost the same we block applications of the rules $(\Box_i \rightarrow)$ and $(\vee \rightarrow)$ and conclude that $KS5_2 \nvdash S$.

3. Specialization of modal rules

With the aim to construct loop-check-free sequent calculus for $S5_2$ let us introduce some specialization of the modal rules $(\rightarrow \Box_i)$ and $(\Box_i \rightarrow)$. At first, a notion of primary sequent is defined.

DEFINITION 1. A sequent S is a *primary* one if $S = \Sigma_1,\ \Box_1\Gamma_1,\ \Box_2\Gamma_2 \rightarrow \Sigma_2,\ \Box_1\Delta_1,\ \Box_2\Delta_2$, where Σ_i $(i \in \{1,2\})$ is empty or consists of propositional symbols, $\Box_i\Gamma_i$ and $\Box_i\Delta_i$ $(i \in \{1,2\})$ is empty or consists of the formulas of the shape $\Box_i A$.

Using invertibility of logical rules we can prove

LEMMA 2 (reduction to primary sequents). *It is possible automatically construct a reduction of a sequent S to a set $\{S_1,\dots,S_m\}$, where S_j $(1 \leqslant j \leqslant m)$ is a primary sequent. Moreover, if $K_1S5_2 \vdash^V S$, where V is an atomic derivation, then $K_1S5_2 \vdash^{V_j} S_j$ $(j \in \{1,\dots,m\})$ and $h(V_j) < h(V)$.*

Let K_1S5_2 be a calculus obtained from the initial calculus $KS5_2$ replacing the modal rules $(\rightarrow \Box_i)$ and $(\Box_i \rightarrow)$ by the following ones:

$$\frac{\Sigma_1,\ \Box_1\Gamma_1,\ \Box_2\Gamma_2 \rightarrow \Sigma_2,\ \Box_1\Delta_1,\ \Box_2\Delta_2,\ A^{\langle i,r\rangle}}{\Sigma_1,\ \Box_1\Gamma_1,\ \Box_2\Gamma_2 \rightarrow \Sigma_2,\ \Box_1\Delta_1,\ \Box_2\Delta_2,\ \Box_i A}\ (\rightarrow \Box_i^p)$$

$$\frac{\Sigma_1,\ A^{\langle i,\alpha\rangle},\ \Box_i A,\ \Box_1\Gamma_1,\ \Box_2\Gamma_2 \rightarrow \Sigma_2,\ \Box_1\Delta_1,\ \Box_2\Delta_2}{\Sigma_1,\ \Box_i A,\ \Box_1\Gamma_1,\ \Box_2\Gamma_2 \rightarrow \Sigma_2,\ \Box_1\Delta_1,\ \Box_2\Delta_2}\ (\Box_i^p \rightarrow),$$

where conclusion of the rule is a primary sequent and Σ_j $(j \in \{1,2\})$ is empty or consists of propositional symbols, moreover, $\Sigma_1 \cap \Sigma_2$ is empty. In the premises of these rules the indices $\langle i,r\rangle$ and $\langle i,\alpha\rangle$ are defined as in the rules $(\rightarrow \Box_i)$ and $(\Box_i \rightarrow)$, correspondingly.

Relying on Lemma 2 we get

LEMMA 3. *$KS5_2 \vdash S$ if and only if $K_1S5_2 \vdash S$.*

4. Loop-check-free sequent calculus for $S5_2$

Along with usual modality $\Box_i$ let us introduce a marked modality $\Box_i^*$ which has the same semantical meaning as non-marked modality $\Box_i$ and serves as a stopping device for a backward application of the rules $(\rightarrow \Box_i)$ and $(\Box_i \rightarrow)$.

Let us introduce operation $*$ defined in the following way:

1. $(P^\gamma)^* = P^\gamma$, where index γ can be empty.

2. $(A \odot B)^* = A^* \odot B^*$, where $\odot \in \{\supset, \wedge, \vee\}$.
3. $(\neg A)^* = \neg A^*$.
4. $(\Box_i A)^* = \Box_i^* A^*$ $(i \in \{1, 2\})$.
5. $\Box_i^{**} A = \Box_i^* A$ $(i \in \{1, 2\})$.

Let $K_1^* S5_2$ be a calculus obtained from the calculus $K_1 S5_2$ replacing the rules $(\to \Box_i^p)$ and $(\Box_i^p \to)$ by the following marked rules:

$$\frac{\Sigma_1, \Box_1^{*\circ}\Gamma_1, \Box_1\Theta_1, \Box_2^{*\circ}\Gamma_2, \Box_2\Theta_2 \to \Sigma_2, \Box_1^*\Pi_1, \Box_1\Delta_1, \Box_2^*\Pi_2, \Box_2\Delta_2, A^{\langle i,r\rangle}}{\Sigma_1, \Box_1^*\Gamma_1, \Box_1\Theta_1, \Box_2^*\Gamma_2, \Box_2\Theta_2 \to \Sigma_2, \Box_1^*\Pi_1, \Box_1\Delta_1, \Box_2^*\Pi_2, \Box_2\Delta_2, \Box_i A} (\to \Box_i^*)$$

$$\frac{\Sigma_1, A^{\langle i,\alpha\rangle}, \Box_i^* A^*, \Box_1^*\Gamma_1, \Box_1\Theta_1, \Box_2^*\Gamma_2, \Box_2\Theta_2 \to \Sigma_2, \Box_1^*\Pi_1, \Box_2^*\Pi_2}{\Sigma_1, \Box_i A, \Box_1^*\Gamma_1, \Box_1\Theta_1, \Box_2^*\Gamma_2, \Box_2\Theta_2 \to \Sigma_2, \Box_1^*\Pi_1, \Box_2^*\Pi_2} (\Box_i^* \to)$$

where Σ_j $(j \in \{1, 2\})$ is empty or consists of propositional symbols, moreover, $\Sigma_1 \cap \Sigma_2$ is empty; in the conclusion of the rules the modality $\Box_i$ in the main formula $\Box_i A$ is not marked; in the premises of the rules the indices $\langle i, r\rangle$ and $\langle i, \alpha\rangle$ are defined as in the rules $(\to \Box_i)$ and $(\Box_i \to)$, correspondingly; in the premise of the rule $(\to \Box_i^*)$, if $i = j$ $(j \in \{1, 2\})$ then $\Box_j^{*\circ}\Gamma_j = \Box_j\Gamma_j$ otherwise $\Box_j^{*\circ}\Gamma_j = \Box_j^*\Gamma_j$.

LEMMA 4. *If $K_1^* S5_2 \vdash^V S$ then $K_1 S5_2 \vdash S$ where S does not contain occurrences of marked modality.*

Proof. Let us replace the marked modalities $\Box_i^*$ $(i \in \{1, 2\})$ in the derivation V by the non-marked ones. As a result we get that each application of marked modal rule transforms into application of non-marked modal rule.

To prove the inverse implication let us introduce marked rules $(\to \Box_i^{*d})$ and $(\Box_i^{*d} \to)$ which are obtained from the rules $(\to \Box_i^*)$ and $(\Box_i^* \to)$ correspondingly by replacing the non-marked main formula $\Box_i A$ by the marked formula $\Box_i^* A$, i.e., the main formula $\Box_i^* A$ is marked.

Let $K_1^{*d} 5_2$ be an auxiliary calculus obtained from the calculus $K_1^* S5_2$ adding the rules $(\to \Box_i^{*d})$, $(\Box_i^{*d} \to)$.

We can prove the following

LEMMA 5. *If $K_1^{*d} S5_2 \vdash S$ then $K_1^* S5_2 \vdash S$, where S does not contain marked modality.*

Using this lemma we get

LEMMA 6. *If $K_1 S5_2 \vdash S$ then $K_1^* S5_2 \vdash S$.*

From Lemmas 4 and 6 we get

LEMMA 7. *$K_1 S5_2 \vdash S$ if and only if $K_1^* S5_2 \vdash S$.*

Relying on Lemmas 3, 7 and Theorem 1 we get

THEOREM 3. *The calculus $K_1^* S5_2$ is sound and complete.*

Using traditional technique we can prove

LEMMA 8. *Let S (S_1) be a conclusion (premise, correspondingly) of any rule of $K_1^* S5_2$. Let $K_1^* S5_2 \vdash^V S$ where V is an atomic derivation of S. Then $K_1^* S5_2 \vdash S_1$.*

A primary sequent S of the shape $\Sigma_1, \Box_1^* \Gamma_1, \Box_2^* \Gamma_2 \rightarrow \Sigma_2, \Box_1^* \Delta_1, \Box_2^* \Delta_2$ is a *critical* one if $\Sigma_1 \cap \Sigma_2$ is empty. A derivation V of a sequent S in $K_1^* S5_2$ is *successful* if each branch of V ends with an axiom. In this case $K_1^* S5_2 \vdash S$. A derivation V of a sequent S in $K_1^* S5_2$ is *unsuccessful* if V contains a branch ending with a critical sequent. In this case $K_1^* S5_2 \nvdash S$.

From Lemma 8 on invertibility of the rules of $K_1^* S5_2$ and shape of these rules we get

THEOREM 4. *$K_1^* S5_2$ is a loop-check-free decidable calculus, i.e., for any sequent S there exists a successful or unsuccessful derivation without loops of the sequent S in $K_1^* S5_2$.*

Example 1. Let us construct a derivation of the sequent $S = \Box_1 P \rightarrow \Box_1 \neg \Box_2 \neg P$ in $K_1^* S5_2$.

$$\dfrac{\dfrac{\dfrac{S^* = P^{\langle 1,\beta|\beta\in\{1\}\rangle}, \Box_1^* P, \Box_2^* \neg P^{\langle 1,1\rangle} \rightarrow P^{\langle 1,1\rangle,\langle 2,\alpha|\alpha\in\{0\}\rangle}}{S' = \Box_1 P, \Box_2^* \neg P^{\langle 1,1\rangle} \rightarrow P^{\langle 1,1\rangle,\langle 2,\alpha|\alpha\in\{0\}\rangle}}(\Box_1^* \rightarrow)}{\Box_1 P, \Box_2 \neg P^{\langle 1,1\rangle} \rightarrow}(\Box_2^* \rightarrow), (\neg \rightarrow)}{S = \Box_1 P \rightarrow \Box_1 \neg \Box_2 \neg P}(\rightarrow \Box_1^*), (\rightarrow \neg)$$

Choosing $\beta = 1$ and $\alpha = 0$ and applying indexation procedure, we get that the sequent S^* is an axiom of the shape $P^{\langle 1,1\rangle}, \Box_1^* P, \Box_2^* \neg P^{\langle 1,1\rangle} \rightarrow P^{\langle 1,1\rangle}$.

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REZIUMĖ

A. Pliuškevičienė. Ciklų tikrinimo eliminavimas idealaus žinojimo logikai

Pateiktas korektiškas ir pilnas sekvencinis skaičiavimas idealaus žinojimo logikai. Naudojant sukonstuotą skaičiavimą gaunama neturinti ciklų išprendžiamoji procedūra. Vietoje istorijų sąvokos įrodymų baigtinumo tikrinimui naudojami indeksai ir žymės.

Raktiniai žodžiai: modalumo logikos, sekvencinis skaičiavimas, išvedimo baigtinumas, ciklų tikrinimas, indeksavimas.