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Termination of derivations for minimal tense logic

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Abstract. It is known that loop checking and backtracking are extensively used in various non-classical logics. An efficient loop checking is obtained using a technique based on histories. In the paper a method for elimination of loop checking in backward proof search for minimal tense logic K_t is proposed. To obtain termination of derivation indices and marks are used instead of history.

Keywords: tense logic, sequent calculus, indexation, termination of derivation, loop-check.

1. Introduction

In the paper minimal tense logic K_t is considered. Minimal tense logic K_t was extensively studied by various authors. In [4] Hilbert-style axiomatization of K_t was described, completeness and finite model property (decidability) were proved. In [5, 3] Gentzen-style calculi with some restricted cut rule for K_t are presented. In [2] a graph-style calculus for K_t and efficient loop-check with histories is described. In [1] a labeled sequent calculus with histories is presented. Since the logic K_t can be regarded as some prototype for temporal logic with past operator and for dynamic logic with inverse operator, this logic is interesting to Computer Science and Artificial Intelligence.

It is known that constructing derivations loop checking and backtracking are extensively used in various non-classical logics. An efficient loop checking is obtained using a technique based on histories. In the paper indexed loop-check-free and cut-free invertible sequent calculus for K_t is presented. Instead of histories marking of modal operators and indexation of propositional variables are used. The proposed technique is based on invertible and decidable sequent calculus which has some similarity with Kanger-style invertible indexed calculus for modal logic $S5$. The soundness, completeness, termination and decidability of constructed calculus are stated.

2. Initial Hilbert-style and Gentzen-style calculi for K_t

A language of K_t consists of a set of propositional symbols, logical symbols $\supset$, $\wedge$, $\vee$, $\neg$ and two modalities $\Box_1$ and $\Box_2$. Formulas are constructed in traditional way from propositional symbols using logical symbols and modalities $\Box_i$ $(i \in \{1, 2\})$. The language does not contain the modality $\Diamond_i$ $(i \in \{1, 2\})$ assuming that $\Diamond_i A = \neg\Box_i\neg A$. The modality $\Box_1$ $(\Box_2)$ may be read as "at all future times" ("at all previous times", correspondingly). Sequent is a formal expression $A_1, \dots, A_k \to B_1, \dots, B_m$, where $A_1, \dots, A_k$ $(B_1, \dots, B_m)$ is a multiset of formulas.

A Hilbert-style calculus HK_t for K_t is obtained from Hilbert-style calculus H for classical propositional logic by adding the following postulates:

$$1_i.\ \Box_i(A \supset B) \wedge \Box_i A \supset \Box_i B \quad (i \in \{1, 2\});$$

$$2.\ A \supset \Box_1 \neg \Box_2 \neg A; \qquad 3.\ A \supset \Box_2 \neg \Box_1 \neg A;$$

and the rule of inference $A/\Box A$ $(\Box)$.

In [4] it is proved that HK_t is sound and complete.

A Gentzen-style calculus GK_t for K_t is obtained from Gentzen-style calculus G for classical propositional logic by adding the following rules:

$$\frac{\Gamma \to A}{\Pi, \Box_i \Gamma \to \Delta, \Box_i A}\ (\Box_i) \quad (i \in \{1, 2\})$$

$$\frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, \Box_1 \neg \Box_2 \neg A}\ (\to \Box_1) \qquad \frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, \Box_2 \neg \Box_1 \neg A}\ (\to \Box_2)$$

$$\frac{\Gamma \to \Delta, \Box_i A;\ \Box_i A, \Gamma \to \Delta}{\Gamma \to \Delta}\ (cut_{\Box_i})$$

THEOREM 1. *A formula A is provable in HK_t if and only if a sequent $\to A$ is provable in GK_t, i.e., the calculus GK_t is sound and complete.*

3. Cut-free sequent calculus for K_t

To obtain cut-free sequent calculus for K_t along with the set of propositional symbols we introduce a set of indexed propositional symbols of the shape P^γ where γ is a list (possibly empty) of the shape $\sigma_1 a_1, \sigma_2 a_2, \dots, \sigma_n a_n$, where $\sigma_i \in \{+, -\}$ and a_i is a parameter.

Let γ be any index, then indexation of any formula is defined as follows:
1. $(P^\gamma)^{\sigma a} = P^{\gamma, \sigma a}$.
2. $(A \odot B)^\gamma = A^\gamma \odot B^\gamma$, where $\odot \in \{\supset, \wedge, \vee\}$.
3. $(\partial A)^\gamma = \partial A^\gamma$, where $\partial \in \{\neg, \Box_1, \Box_2\}$.

It is assumed that $P^{\gamma, \sigma a, \lambda a, \gamma_1} = P^{\gamma, \gamma_1}$, where $\sigma \in \{+, -\}$, $\lambda \in \{+, -\}$ and $\sigma \neq \lambda$; γ and γ_1 are arbitrary indices.

A cut-free sequent calculus $G_1 K_t$ for K_t is obtained from GK_t replacing the modal rules $(\Box_i)$, $(\to \Box_i)$ and $(cut_{\Box_i})$ where $i \in \{1, 2\}$ by the following rules:

$$\frac{\Gamma \to \Delta, A^{\sigma b}}{\Gamma \to \Delta, \Box_i A}\ (\to \Box_i) \qquad \frac{A^{\sigma \alpha}, \Box_i A, \Gamma \to \Delta}{\Box_i A, \Gamma \to \Delta}\ (\Box_i \to),$$

where $i \in \{1, 2\}$ and if $i = 1$ then $\sigma = +$ else $\sigma = -$; in the rule $(\to \Box_i)$ b does not enter in the conclusion; in the rule $(\Box_i \to)$ α is metaparameter such that if the conclusion does not contain any indices, then value of α is an arbitrary parameter otherwise value of α is chosen from parameters entering in the conclusion.

THEOREM 2. *The cut rule is admissible in G_1K_t.*

THEOREM 3. *All rules of the calculus G_1K_t are invertible in G_1K_t.*

Relying on Theorem 1 we get

THEOREM 4. *The calculus G_1K_t is a conservative extension of GK_t, i.e., the calculus G_1K_t is sound and complete.*

Remark 1. The duplication of the main formula in the premise of the rule $(\Box_i \rightarrow)$ is necessary. The calculus $G_1^dK_t$ obtained from the calculus G_1K_t replacing the rule $(\Box_i \rightarrow)$ by the rule $(\Box_i^d \rightarrow)$ in premise of which the main formula is not preserved is incomplete. Let $S = \Box_1(P \wedge \neg\Box_1 P) \rightarrow$. Then it is easy to verify that $G_1K_t \vdash S$ but $G_1^dK_t \nvdash S$. The sequent S is derivable in the defined in [3] calculus GK_t using cut rule.

4. Loop-check-free sequent calculus for K_t

From the shape of the rule $(\Box_i \rightarrow)$ it follows that to ensure termination of derivations in backward proof search a loop-check (with histories or without them) is necessary. With the aim to eliminate a loop-check instead of the rules $(\Box_i \rightarrow)$, $(\rightarrow \Box_i)$ marked modal rules are used. Along with non-marked modalities $\Box_i$ marked modalities $\Box_i^*$ are used. The marked modalities have the same semantical meaning as non-marked modalities $\Box_i$ and serves as a stopping device for a backward application of the rules $(\rightarrow \Box_i)$ and $(\Box_i \rightarrow)$.

To define the marked modal rules let us introduce operation $*$ which is applied to any formula and defined in the following way:

1. $(P^\gamma)^* = P^\gamma$, where index γ can be empty.
2. $(A \odot B)^* = A^* \odot B^*$, where $\odot \in \{\supset, \wedge, \vee\}$.
3. $(\neg A)^* = \neg A^*$.
4. $(\Box_i A)^* = \Box_i^* A^*$ $(i \in \{1, 2\})$.

It is assumed that $\Box_i^{**} = \Box_i^*$ $(i \in \{1, 2\})$.

Let $G_1^*K_t$ be a calculus obtained from the calculus G_1K_t replacing the modal rules $(\rightarrow \Box_i)$ and $(\Box_i \rightarrow)$ by the following marked rules:

$$\frac{\Sigma_1, \Box_1^{*\circ}\Gamma_1, \Box_1\Theta_1, \Box_2^{*\circ}\Gamma_2, \Box_2\Theta_2 \rightarrow \Sigma_2, \Box_1^*\Pi_1, \Box_1\Delta_1, \Box_2^*\Pi_2, \Box_2\Delta_2, A^{\sigma b}}{\Sigma_1, \Box_1^*\Gamma_1, \Box_1\Theta_1, \Box_2^*\Gamma_2, \Box_2\Theta_2 \rightarrow \Sigma_2, \Box_1^*\Pi_1, \Box_1\Delta_1, \Box_2^*\Pi_2, \Box_2\Delta_2, \Box_i A} (\rightarrow \Box_i^*)$$

$$\frac{\Sigma_1, A^{\sigma\alpha}, \Box_i^* A^*, \Box_1^*\Gamma_1, \Box_1\Theta_1, \Box_2^*\Gamma_2, \Box_2\Theta_2 \rightarrow \Sigma_2, \Box_1^*\Delta_1, \Box_2^*\Delta_2}{\Sigma_1, \Box_i A, \Box_1^*\Gamma_1, \Box_1\Theta_1, \Box_2^*\Gamma_2, \Box_2\Theta_2 \rightarrow \Sigma_2, \Box_1^*\Delta_1, \Box_2^*\Delta_2} (\Box_i^* \rightarrow)$$

where Σ_j $(j \in \{1, 2\})$ is empty or consists of propositional symbols, moreover, $\Sigma_1 \cap \Sigma_2$ is empty; in the conclusion of the rules the modality $\Box_i$ in the main formula $\Box_i A$ is not marked; in the premises of the rules the indices σb and $\sigma\alpha$ are defined as in the rules $(\rightarrow \Box_i)$ and $(\Box_i \rightarrow)$, correspondingly; in the premise of the rule $(\rightarrow \Box_i^*)$, if $i = j$ $(j \in \{1, 2\})$ then $\Box_j^{*\circ}\Gamma_j = \Box_j\Gamma_j$ otherwise $\Box_j^{*\circ}\Gamma_j = \Box_j^*\Gamma_j$.

Using invertibility of the rules of $G_1^*K_t$ we can prove

THEOREM 5. *$G_1K_t \vdash S$ if and only if $G_1^*K_t \vdash S$, i.e., the calculus $G_1^*K_t$ is sound and complete.*

A sequent S of the shape $\Sigma_1, \Box_1^*\Gamma_1, \Box_2^*\Gamma_2 \to \Sigma_2, \Box_1^*\Delta_1, \Box_2^*\Delta_2$ is a *final* one if $\Sigma_1 \cap \Sigma_2$ is empty. A derivation V of a sequent S in $G_1^*K_t$ is *successful* if each branch of V ends with an axiom. In this case $G_1^*K_t \vdash S$. A derivation V of a sequent S in $G_1^*K_t$ is *unsuccessful* if V contains a branch ending with a final sequent. In this case $G_1^*K_t \nvdash S$.

From the shape of the rules of $G_1^*K_t$ we get

THEOREM 6. *Any backward derivation in calculus $G_1^*K_t$ of a sequent S terminates.*

Example 1. (a) Let S be a sequent $\Box_1 P \to \Box_1\neg\Box_2\neg\Box_1 P$ from Example in [5]. Then a successful derivation of S in $G_1^*K_t$ is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{S^* = \Box_1^* P,\, P^{+\alpha|\alpha\in\{b,c\}},\, \Box_2^*\neg\Box_1^* P^{+b} \to P^{+c}}{\Box_1 P, \Box_2^*\neg\Box_1^* P^{+b} \to P^{+c}}(\Box_1^* \to)}{S' = \Box_1 P, \Box_2^*\neg\Box_1^* P^{+b} \to \Box_1 P}(\to \Box_1^*)}{\Box_1 P, \Box_2\neg\Box_1 P^{+b} \to}(\Box_2^* \to), (\neg \to)}{S = \Box_1 P \to \Box_1\neg\Box_2\neg\Box_1 P}(\to \Box_1^*), (\to \neg)$$

Choosing $\alpha = c$ we get that the sequent S^* is an axiom of the shape $\Box_1^* P, P^{+c}, \Box_2^*\neg\Box_1^* P^{+b} \to P^{+c}$. It is obvious that in the sequent S' the formula $\Box_1 P$ in the succedent is obtained from $\Box_1(P^{+b})^{-b}$ using indexation procedure.

(b) Let us construct a derivation of the sequent $P \to \neg\Box_2\neg\Box_1 P$ from Example in [5].

$$\dfrac{\dfrac{\dfrac{\dfrac{S^* = P, \Box_2^*\neg\Box_1^* P \to P^{-a,+b}}{P, \Box_2^*\neg\Box_1^* P \to \Box_1 P^{-a}}(\to \Box_1^*)}{P, \Box_2^*\neg\Box_1^* P, \neg\Box_1 P^{-a} \to}(\neg \to)}{P, \Box_2\neg\Box_1 P \to}(\Box_2^* \to)}{S = P \to \neg\Box_2\neg\Box_1 P}(\to \neg)$$

Since sequent S^* is a final sequent we get $G_1^*K_t \nvdash S$. It is obvious that in the sequent S^* the formula $P^{-a,+b}$ in the succedent is obtained from $(P^{-a})^{+b}$ using indexation procedure.

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REZIUMĖ

R. Pliuškevičius. Įrodymų baigtinumas minimaliai laiko logikai

Straipsnyje pateiktas korektiškas ir pilnas beciklis sekvencinis skaičiavimas minimaliai laiko logikai. Įrodymų baigtinumo tikrinimui naudojami indeksai ir žymės.

Raktiniai žodžiai: laiko logika, sekvencinis skaičiavimas, indeksavimas, išvedimo baigtinumas, ciklų tikrinimas.