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A method of marks and indices for linear modal logic

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Abstract. In the paper a method to check termination of history-free proof for linear modal logic $S4.3$ is proposed. This method improves the method proposed by the authors for modal logic $S4$. Analogously as for $S4$, instead of history we use marks and indices that allow us to eliminate loop checking. The method proposed in this paper specifies some kind of formulas which allow us to check termination of derivations in more effective way in comparison with $S4$.

Keywords: modal logics, sequent calculus, termination, invertible rules.

1. Introduction

In [3] the notion of history to ensure termination of derivations in some non-classical logics was introduced. The history allows us to achieve efficient loop checking by means of an information about previous parts of a derivation. The history based method nowadays is widely used constructing derivations in non-classical logics. In [4] a method called marks and indices method (denoted $M\&I$) for modal logic $S4$ was proposed. In $M\&I$, instead of history marks and indices that allow us to eliminate loop checking are used. In the present paper an improved version of $M\&I$ method for linear modal logic $S4.3$ is described. The logic $S4.3$ is obtained from modal logic $S4$ adding the linearity axiom $\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$. The logic $S4.3$ is interpreted as discrete linear time logic. The aim of this paper is to construct an invertible sequent calculus for modal logic $S4.3$ without loop checking changing and extending the technique from [4].

2. Invertible calculus with specialized reflexivity rule

Formulas in the considered calculus are constructed in traditional way from propositional symbols using the classical logical connectives and the necessity modality $\Box$. Along with modality $\Box$ a marked modality $\Box^*$ is introduced. This marked modality has the same semantical meaning as non-marked modality $\Box$ and serves as a device to restrict a backward application of reflexivity rule. A formula of the shape $\Box A$ is called a modal one. The language considered does not contain the modality $\Diamond$ assuming that $\Diamond A = \neg\Box\neg A$. We consider sequents, i.e., formal expressions $A_1, \ldots, A_k \to B_1, \ldots, B_m$, where $A_1, \ldots, A_k$ $(B_1, \ldots, B_m)$ is a multiset of formulas. For simplicity we consider sequents not containing branching formulas (see, e.g., [2]).

A sequent S is a *primary* one if (1) S has the shape $\Sigma_1, \Box^*\Gamma \to \Sigma_2, \Box\Delta$, where Σ_i ($i \in \{1,2\}$) is empty or consists of propositional symbols; $\Box^*\Gamma$ is empty or consists of formulas of the shape $\Box^*M$; $\Box\Delta$ is empty or consists of formulas of the shape $\Box M$, and (2) antecedent and/or succedent of the sequent does not contain several occurrences of the same formula.

Cut-free sequent calculus with specialized reflexivity rule $GS4.3$ for modal logic $S4.3$ is defined by the following postulates (see, e.g., [1]):

Axiom: $\Gamma, P \to \Delta, P$

where multiset Γ is permitted to contain some formulae of the shape $\Box^*B$, i.e., modality $\Box$ *can be marked*.

Logical rules:

Traditional invertible rules for logical connectives $\supset, \wedge, \vee, \neg$.

Modal rules:

$$\frac{A, \Box^*A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box^* \to),$$

where in the conclusion of the rule the outmost occurrence of modality $\Box$ in the formula $\Box A$ is *not marked* but some of modal formulas from Γ *can be marked*. The rule $(\Box^* \to)$ is called *reflexivity rule* because it corresponds to reflexivity axiom $\Box A \supset A$.

$$\frac{\Gamma, \Box^*\Gamma \to A_1, \Box A_2, \dots, \Box A_n; \dots; \Gamma, \Box^*\Gamma \to \Box A_1, \dots, \Box A_{n-1}, A_n}{\Sigma_1, \Box^*\Gamma \to \Sigma_2, \Box A_1, \dots, \Box A_n}(\Box),$$

where the conclusion of the rule is a primary sequent such that $\Sigma_1 \cap \Sigma_2$ is empty. The rule $(\Box)$ is called *linearity rule* because it corresponds to linearity axiom $\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$.

From [1] it follows that the calculus $GS4.3$ is sound and complete.

Using traditional proof-theoretical methods we get that each rule of the calculus $GS4.3$ is invertible in $GS4.3$.

3. Loop-check-free calculus for *S4.3*

To construct backward proof search without loop checking a notion of an indexed modality is introduced and sequents containing occurrences of the indexed modality are considered. Let us introduce the following indexation technique.

A positive occurrence of modality $\Box$ in a sequent S is a *special* one if it occurs within the scope of a negative occurrence of modality $\Box$ in S. A special occurrence α of modality $\Box$ in a sequent is an *isolated* one if within the scope of α there is a negative occurrence of modality $\Box$. We distinguish two sorts of isolated occurrences of the modality. An isolated occurrence α of modality $\Box$ in a sequent is *strongly special* if within the scope of α there are no isolated occurrences of $\Box$. A special occurrence of $\Box$ which is not strongly special is *simply special*.

Let us introduce two sorts of indexes used only for special occurrences of modality, namely, index i where $i \in \{1, \dots, n\}$ and n is the number of simply special occurrences of modality $\Box$ in a sequent, and index $\circ k$, where $k \in \{1, \dots, m\}$ and m is the number of strongly special occurrences of modality $\Box$ in a sequent. The modality $\Box^\sigma$ ($\sigma \in \{i, \circ k\}$) is an *indexed modality*.

For example, let $S = \Box\neg\Box(\neg\Box Q \vee (\Box\neg\Box\neg P \vee \Box\Box\neg\Box\neg\Box P)) \rightarrow$, then S_{ind} has the shape $\Box\neg\Box^1(\neg\Box Q \vee (\Box^{\circ 1}\neg\Box\neg P \vee \Box^2\Box^{\circ 2}\neg\Box\neg\Box^3 P)) \rightarrow$; the occurrences of $\Box^1, \Box^2$, $\Box^{\circ 1}$, $\Box^{\circ 2}$ in S_{ind} are isolated ones, and the occurrence of $\Box^3$ in S_{ind} is not isolated one; there are no isolated occurrences of the modality within the scope of the occurrences of $\Box^{\circ 1}$, $\Box^{\circ 2}$, but there are isolated occurrences of the modality within the scope of the occurrences of $\Box^1, \Box^2$.

Along with the marked modality $\Box^*$ (introduced in the previous section and used only for negative occurrences of modality) let us introduce one more marked modality, namely, $\Box^+$. The marked modality $\Box^+$ serves as a stopping device for a backward application of the linearity rules. The marked and indexed modalities have the same semantical meaning as non-marked and non- indexed modality $\Box$. Let A be a formula from a sequent S, then an *indexed formula* A_{ind} is a formula obtained from A by replacing any simply (strongly) special occurrence of $\Box$ in A by the indexed modality $\Box^i$ ($\Box^{\circ k}$, correspondingly) in such a way that different special occurrences of $\Box$ get different indices. Let S be a sequent, then an *indexed sequent* S_{ind} is a sequent obtained from S by replacing every formula in S by appropriate indexed formula in such a way that different special occurrences of $\Box$ in an indexed sequent S_{ind} get different indices.

A simply special occurrence of modality $\Box$, i.e., indexed modality of the shape $\Box^i$, in S_{ind} is *dependent* if within the scope of $\Box^i$ there is at least one occurrence of some indexed modality $\Box^\sigma$ ($\sigma \in \{i, \circ k\}$). In opposite case the occurrence of $\Box^i$ in S_{ind} is *independent*.

For example, let $S = \Box\neg\Box\Box(\Box P \vee \Box\Box\neg\Box P) \rightarrow$, then S_{ind} has the shape $\Box\neg\Box^1\Box^2(\Box^3 P \vee \Box^4\Box^{\circ 1}\neg\Box P) \rightarrow$; the occurrence of $\Box^3$ in S_{ind} is *independent* one, and occurrences of $\Box^1$, $\Box^2$, $\Box^4$ in S_{ind} are *dependent* ones.

Let us introduce an *operation* $\sigma+$ ($\sigma \in \{i, \circ k\}$). Let A be any indexed formula from an indexed sequent S_{ind}. Then application of the operation $\sigma+$ to A is denoted as $A^{\sigma+}$ and the result of this application is a formula obtained from A by replacing the occurrence of $\Box^\sigma$ in A by marked modality $\Box^+$. If A does not contain occurrences of $\Box^\sigma$ then $A^{\sigma+} = A$. The notation $\Gamma^{\sigma+}$ means $A_1^{\sigma+}, \ldots, A_k^{\sigma+}$, where $k \geqslant 1$ and Γ is a sequence of indexed formulas $A_1, \ldots, A_k$.

Let us note that only positive occurrences of the modality $\Box$ may get indexes or the mark $+$ and only negative occurrences of $\Box$ may get the mark $*$.

Let us introduce the following notions which allow us to check termination of derivations in more effective way in comparison with checking for $S4$ described in [4].

Let B be a formula entering in a sequent S. A subformula of B is a modal one if it has the shape $\Box^\mu M$, where $\mu \in \{\varnothing, i, \circ k, +, *\}$.

A modal formula B is a *passive formula* if

- B occurs in a sequent S positively and has the shape $\Box^{i_1} \ldots \Box^{i_n} M$ ($n \geqslant 1$), where M is a formula containing at least one occurrence of index-free modality (probably, marked modality) and does not contain any occurrences of indexed modality; B is called a *passive formula of the first type*;
- B occurs in a sequent S positively, has the shape $\Box^{\tau_1} \ldots \Box^{\tau_n} M$ ($n \geqslant 1$), $\tau_j \in \{i, +\}$, $j \in \{1, \ldots, n\}$ and there exists j such that $\tau_j = +$; B is called a *passive formula of the second type*;

- B occurs in a sequent S negatively and has the shape $\Box^* \overbrace{\Box \ldots \Box}^{m\ times} M$ $(m \geqslant 0)$, where M is a formula composed of the first and/or the second kind passive formulas using logical symbols; B is called an *passive formula of the third type*.

Any modal formula that is not passive one is *active formula*.

For example, let S be a sequent $\Box^*\neg\Box^1 P, \Box^*\neg\Box^2\Box^+ Q \to \Box^1 P, \Box^2\Box^+ Q, \Box^4(\Box^+ P_1 \vee \Box^+ Q_1), \Box R$. Then the formula $\Box^4(\Box^+ P_1 \vee \Box^+ Q_1)$ is the passive formula of the first kind; the formula $\Box^2\Box^+ Q$ is the passive formula of the second kind; the formula $\Box^*\neg\Box^2\Box^+ Q$ is the passive formula of the third kind. Formulas $\Box^*\neg\Box^1 P$, $\Box^1 P$ and $\Box R$ are active formulas.

An indexed sequent S is a *primary* one if (1) S has the shape $\Sigma_1, \Box^*\Gamma \to \Sigma_2, \tilde{\Box}\Delta$, where Σ_i $(i \in \{1, 2\})$ is empty or consists of propositional symbols; $\Box^*\Gamma$ is empty or consists of formulas of the shape $\Box^* M$; $\tilde{\Box}\Delta$ is empty or consists of formulas of the shape $\Box^\mu M$ $(\mu \in \{\varnothing, i, \circ k, +\}$, and for any formulas A and B, if $\Box^\sigma A \in \tilde{\Box}\Delta$ $(\sigma \in \{i, \circ k\})$ and $A \neq B$ then for the same index σ, $\Box^\sigma B \notin \tilde{\Box}\Delta$, and (2) antecedent and/or succedent of the sequent S does not contain several occurrences of the same formula.

Taking into account the introduced notions of active and passive formulas let us specify the shape of the succedent part of a primary sequent. Namely, the part $\tilde{\Box}\Delta$ of primary sequent has the shape $\Box\nabla, \Box^\lambda\Theta, \Box^+\Pi$ where $\Box\nabla$ is empty or consists of active non-indexed formulas, $\Box^\lambda\Theta$ is empty or consists of active indexed formulas, and $\Box^+\Pi$ is empty or consists of the first and/or the second kind passive formulas.

Let $G_1S4.3$ be a calculus obtained from the calculus $GS4.3$ replacing the rule $(\Box)$ by the following linearity rule:

$$\frac{S_1; \ \ldots \ S_j; \ \ldots \ S_n}{\Sigma_1, \Box^*\Gamma \to \Sigma_2, \Box^{\sigma_1} A_1, \ldots, \Box^{\sigma_j} A_j, \ldots, \Box^{\sigma_n} A_n, \Box^+\Delta} \, (\Box_p^\sigma),$$

where the conclusion is a primary sequent such that $\Sigma_1 \cap \Sigma_2$ is empty, $\Box^+\Delta$ is empty or consists of passive formulas of the first or second type; $\Box^{\sigma_1} A_1, \ldots, \Box^{\sigma_j} A_j, \ldots, \Box^{\sigma_n} A_n$, where $\sigma_j \in \{\varnothing, i, \circ k\}$ $(1 \leqslant j \leqslant n)$, consists of active formulas; σ in the notation of the rule $(\Box_p^\sigma)$ denotes the sequence $\sigma_1^*, \ldots, \sigma_j^*, \ldots, \sigma_n^*$, where $\sigma_j^* \in \{\varnothing, \sigma_j, \sigma_j+\}$.

For every j $(j \in \{1, \ldots, n\})$, the shape of the jth premise of this rule and the meaning of σ_j^* in σ depend on the shape of the jth main formula $\Box^{\sigma_j} A_j$ in the conclusion of this rule. For the sake of simplicity, we can imagine that each premise of the rule $(\Box_p^\sigma)$ is obtained applying one-in-three following rules depending on the shape of the main formula:

Non-indexed rule:

$$\frac{\Gamma_1, \Box^*\Gamma_1 \to \Box\nabla, \Box^\lambda\Theta, A}{\Sigma_1, \Box^*\Gamma \to \Sigma_2, \Box\nabla, \Box^\lambda\Theta, \Box^+\Pi, \Box A} \, (\Box_p),$$

Weak indexed rule:

$$\frac{\Gamma_1^{i+}, \Box^*\Gamma_1 \to \Box\nabla, \Box^\lambda\Theta, A}{\Sigma_1, \Box^*\Gamma \to \Sigma_2, \Box\nabla, \Box^\lambda\Theta, \Box^+\Pi, \Box^i A} \, (\Box_p^i),$$

where Γ in the conclusion of this rule contains a dependent occurrence of $\Box^i$ or contains an occurrence of $\Box^{\circ k}$ for some k.

Strong indexed rules:

$$\frac{\Gamma_1^{\lambda+}, \Box^*\Gamma_1^{\lambda+} \to \Box\nabla, \Box^\lambda\Theta, A}{\Sigma_1, \Box^*\Gamma \to \Sigma_2, \Box\nabla, \Box^\lambda\Theta, \Box^+\Pi, \Box^\lambda A}(\Box_p^{\lambda+}),$$

where $\lambda \in \{i, \circ k\}$ and if $\lambda = i$ then Γ in the conclusion of this rule contains an independent occurrence of $\Box^i$ and does not contain an occurrence of $\Box^{\circ k}$ for some k, i.e., conditions indicated in the rule $(\Box_p^i)$ does not hold.

It is important that, as it follows from the shape of the linearity rule $(\Box_p^\sigma)$, this rule satisfies the following conditions:

- the passive formula can not be the main formula of the linearity rule $(\Box_p^\sigma)$ and passive formulas entering in the conclusion of the rule are not preserved in any premise.
- if the jth main formula of the linearity rule $(\Box_p^\sigma)$ is an indexed formula $\Box^{\sigma_j} A_j$ such that $\sigma_j = i$ (but not $\sigma_j = \circ i$) and Γ in the conclusion of this rule contains a dependent occurrence of $\Box^i$ or contains an occurrence of $\Box^{\circ k}$ for some k, then in the premise S_j the operation σ_j+ is not applied to Γ in $\Box^*\Gamma$.

Example 1. (a) Let S be the indexed sequent of the shape
$\Box^*\neg\Box^1\Box^{\circ 1}(\Box P \supset Q) \to \Box^1\Box^{\circ 1}(\Box P \supset Q), \Box\Box Q.$

Backward applying $(\Box_p^{1,\varnothing})$ to S we get two premises:
$S_1 = \neg\Box^+\Box^{\circ 1}(\Box P \supset Q), \Box^*\neg\Box^1\Box^{\circ 1}(\Box P \supset Q) \to \Box^{\circ 1}(\Box P \supset Q), \Box\Box Q;$
$S_2 = \neg\Box^1\Box^{\circ 1}(\Box P \supset Q), \Box^*\neg\Box^1\Box^{\circ 1}(\Box P \supset Q) \to \Box^1\Box^{\circ 1}(\Box P \supset Q), \Box Q.$
The sequent S_1 is the weak indexed premise and S_2 is the non-indexed premise.

(b) Let S be an indexed sequent of the shape
$\Box^*\neg\Box^1\Box^+(P \vee Q), \Box^*\neg\Box^3(R \supset R) \to \Box^1\Box^+(P \vee Q), \Box^3(R \supset R).$

Since $\Box^*\neg\Box^1\Box^+(P \vee Q)$ is the passive formula of the third type and $\Box^1\Box^+(P \vee Q)$ is the passive formula of the second type, backward applying $(\Box_p^{3+})$ to S we get the strong indexed premise $\neg\Box^+(R \supset R), \Box^*\neg\Box^+(R \supset R) \to (R \supset R)$.

From the shape of the linearity rule $(\Box_p^\sigma)$ it follows that there is the one way to construct the premises of this rule. From this fact we get that the rule $(\Box_p^\sigma)$ is invertible.

A primary sequent of the shape $\Sigma_1, \Box^*\Gamma \to \Sigma_2, \Box^+\Delta$, where $\Sigma_1 \cap \Sigma_2$ is empty and $\Box^*\Gamma$ $(\Box^+\Delta)$ is empty or consist of formulas of the shape $\Box^*M$ (passive formulas of the first and/or second type, correspondingly), is a *final* one. It is impossible to apply any rule to a final sequent.

A derivation V of a sequent S in the calculus $G_1S4.3$ is a *successful* one, if *each* branch of V ends with an axiom. In this case a sequent S is derivable in $G_1S4.3$. A derivation V of S in the calculus $G_1S4.3$ is an *unsuccessful* one if V contains a branch ending with a final sequent. In this case a sequent S is non-derivable.

Let us note that in calculus $G_1S4.3$ derivation of indexed sequent is constructed and indexed end-sequent S_{ind} of a derivation is obtained from arbitrary sequent S

which does not contain any indices and marks. Thus, end-sequent S_{ind} does not contain marked modalities $\Box^*$, $\Box^+$.

Since using invertibility of the rules of $G_1S4.3$ and technique from [4] we can prove that the calculi $GS4.3$ and $G_1S4.3$ are equivalent, we get

THEOREM 1. *The calculus $G_1S4.3$ is sound and complete.*

Analogously as in [4] we can show that complexity of each sequent constructing backward derivation of any indexed sequent S in $G_1S4.3$ decreases. Thus, backward proof search in $G_1S4.3$ terminates.

References

1. R. Goré. Cut-free sequent and tableau systems for propositional Diodorean modal logics. *Technical Report No. 288*, University of Cambridge, Computer Lab., 1993.
2. J. Halpern, Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.
3. A. Heuerding, M. Seyfried, H. Zimmermann. Efficient loop-check for backward proof search in some non-classical propositional logics. In: *Lecture Notes in Computer Science*, Vol. 1071, 1996, pp. 210–225.
4. R. Pliuškevičius, A. Pliuškevičienė. A new method to obtain termination in backward proof search for modal logic *S*4. *Journal of Logic and Computation*, doi: 10.1093/logcom/exn071, 2008.

REZIUMĖ

R. Pliuškevičius, A. Pliuškevičienė. Indeksų ir žymių metodas tiesinei modalumo logikai

Straipsnyje yra pateikiamas indeksų ir žymių metodas tiesinei modalumo logikai. Šis metodas išplečia autorių pasiūlytą metodą modalumo logikai *S*4. Metodo esmė – įrodymų baigtinumo tikrinimas nenaudojant istorijų sąvokos.

Raktiniai žodžiai: modalumo logikos, sekvencinis skaičiavimas, baigtinumas, apverčiamos taisyklės.