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Poisson-type approximation for sums of 1-dependent indicators

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Abstract. The sum of 1-dependent indicators is approximated by compound Poisson type distribution. Estimates are obtained for the uniform Kolmogorov and local metrics.

Keywords: m-dependent random variables, compound Poisson distribution,uniform Kolmogorov metric, local metric.

Let X_j, $j = 1, 2, \dots, n$ be a triangular array of 1-dependent identically distributed three-point random variables, $P(X_j = 1) = p_1$, $P(X_j = -1) = p_{-1}$, $P(X_j = 0) = 1 - p_1 - p_{-1}$. We denote the distribution and characteristic function of $S_n = X_1 + X_2 + \dots + X_n$ by F_n and $\widehat{F}_n(t)$, respectively. Let $x = \mathrm{e}^{\mathrm{i}t} - 1$, $x_{-1} = \bar{x} = \mathrm{e}^{-\mathrm{i}t} - 1$, $p = p_{-1} + p_1$,

$$a(j_1, j_2) = P(X_1 = j_1, X_2 = j_2) - P(X_1 = j_1)P(X_2 = j_2),$$

$$a(j_1, j_2, j_3) = P(X_1 = j_1, X_2 = j_2, X_3 = j_3) - P(X_1 = j_1)P(X_2 = j_2)P(X_3 = j_3),$$

$$b_j = \mathrm{E}(\mathrm{e}^{\mathrm{i}tX_1} - 1)\cdots(\mathrm{e}^{\mathrm{i}tX_j} - 1),$$

$$A_j = \widehat{\mathrm{E}}(\mathrm{e}^{\mathrm{i}tX_1} - 1)\cdots(\mathrm{e}^{\mathrm{i}tX_j} - 1) := b_j - \sum_{k=1}^{j-1} A_k b_{j-k}, \quad A_1 = p_1 x + p_{-1}\bar{x},$$

$$R_1 = \bigl|a(-1,-1) - a(-1,1) - a(1,-1) + a(1,1)\bigr| + p|p_1 - p_{-1}|,$$

$$R_2 = \bigl|a(-1,1) - 2a(1,1) + a(1,-1)\bigr| + \sum_{j,k\in\{-1,1\}} \bigl|a(j,k,-1) - a(j,k,1)\bigr|,$$

$$R_3 = \sum_{j,k\in\{-1,1\}} \bigl|a(j,k)\bigr| + p^2.$$

We use $C_1, C_2, \dots$ to denote positive absolute constants and symbol θ for all (possibly different) quantities, satisfying $|\theta| \leqslant 1$.

Our goal is to investigate the closeness of F_n to its accompanying compound Poisson law. More precisely, let D^n be a compound Poisson distribution with the following characteristic function:

$$\widehat{D}^n(t) = \exp\{nA_1\} = \exp\{np_1 x + np_{-1}\bar{x}\}.$$

The closeness is estimated in the uniform Kolmogorov and local metrics.

THEOREM 1. *Let*

$$\sum_{j,k\in\{-1,1\}} \left|a(j,k)\right|/p + 90\sqrt{p} \leqslant 1/3. \tag{1}$$

Then, for all $n = 1, 2, \dots,$

$$\sup_x \left|F_n\{(-\infty, x]\} - D^n\{(-\infty, x]\}\right| \leqslant C_1 n R_1 \min\left(1, \frac{1}{np}\right) + C_2 n R_2 \min\left(1, \frac{1}{np\sqrt{np}}\right) + C_3 n R_3 \min\left(1, \frac{1}{(np)^2}\right) \tag{2}$$

and

$$\sup_x \left|F_n\{x\} - D^n\{x\}\right| \leqslant C_4 n R_1 \min\left(1, \frac{1}{np\sqrt{np}}\right) + C_5 n R_2 \min\left(1, \frac{1}{(np)^2}\right) + C_6 n R_3 \min\left(1, \frac{1}{(np)^2\sqrt{np}}\right). \tag{3}$$

Remarks

1. We are unaware about any Poisson-type approximation result for dependent random variable, where symmetry is taken into account.
2. Numerous Poisson-type approximations are obtained via the Stein method. However, the Stein method is applicable to non-negative random variables only. Thus, it can not be applied in our case.
3. Condition (1) is a technical one and quite probably can be improved. It is only marginally better than $p = o(1)$, $a(j,k) = o(p)$. Formally, it allows for p to be a (very) small absolute constant.
4. If $X_1, X_2, \dots, X_n$ are symmetric independent random variables, then the right-hand-side in (2) becomes $C_7 n^{-1}$. This is consistent with known facts about Poisson approximation to symmetric three point distributions, see [2].
5. It is not difficult to construct an example of dependent array, which satisfies (1). Let $\xi_1, \xi_2, \dots$ be symmetric i.i.d. r.v., having distribution $P(\xi_1 = 1) = P(\xi_1 = -1) = \alpha$, $P(\xi_1 = 0) = 1 - 2\alpha$. Let $X_1 = \xi_1\xi_2$, $X_2 = \xi_2\xi_3$, etc. If $\alpha = o(1)$, then (1) holds and the accuracy of approximation in (2) is $O(n\alpha^3 \wedge (n\alpha)^{-1})$.

For the proof of Theorem we need auxiliary results. Let $z = \mathrm{i}t$ and $\varphi_1 = \varphi_1(z) = \mathrm{E}\mathrm{e}^{zX_1}$,

$$\varphi_k = \frac{\mathrm{E}\mathrm{e}^{zS_k}}{\mathrm{E}\mathrm{e}^{zS_{k-1}}}, \qquad w_n(\mathrm{i}t) := \sqrt{\mathrm{E}\left|\mathrm{e}^{\mathrm{i}tX_j} - 1\right|^2}, \quad k = 2, 3, \dots, n.$$

For a sequence of arbitrary complex-valued random variables $Y_1, Y_2, \dots$ let $\widehat{\mathrm{E}}Y_1 = \mathrm{E}Y_1$ and

$$\widehat{\mathrm{E}}Y_1Y_2\dots Y_k := \mathrm{E}Y_1Y_2\dots Y_k - \sum_{j=1}^{k-1}\widehat{\mathrm{E}}Y_1\dots Y_j\mathrm{E}Y_{j+1}\dots Y_k, \quad k \geqslant 2.$$

The following lemma follows from Lemma 3.1 and Lemma 3.2 in [1].

LEMMA 1. *Let* (1) *hold. Then, for $k = 1, 2, \dots n$, and all real t*

$$\varphi_k = \mathrm{E}e^{zX_k} + \sum_{j=1}^{k-1}\frac{\widehat{\mathrm{E}}(e^{zX_j}-1)(e^{zX_{j+1}}-1)\dots(e^{zX_k}-1)}{\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}},$$

$$|\varphi_k - 1| \leqslant \left|\mathrm{E}e^{zX_k}-1\right| + 2\sqrt{\mathrm{E}\left|e^{zX_{k-1}}-1\right|^2\mathrm{E}\left|e^{zX_k}-1\right|^2}/(1-4w_n(z))$$
$$\leqslant 13p\left|e^z-1\right| \leqslant 1/5,$$

$$\left|\ln\widehat{F}_n(t) - nA_1\right| \leqslant np\left(\frac{\sum_{j,k\in\{-1,1\}}\left|a(j,k)\right|}{p} + 90\sqrt{p}\right)|x|^2.$$

LEMMA 2. *Let* (1) *be satisfied. Then, for all $|t| \leqslant \pi$,*

$$\max\left\{\left|\widehat{F}_n(t)\right|, \left|\widehat{D}^n(t)\right|\right\} \leqslant \exp\left\{-C_8np\sin^2(t/2)\right\}.$$

Proof. Note that

$$\left|\widehat{F}_n(t)\right| \leqslant \left|\exp\{nA_1\}\right|\exp\left\{\left|\ln\widehat{F}_n(t) - nA_1\right|\right\}$$

and apply Lemma 1. The estimate for $\widehat{D}^n(t)$ follows directly from its definition and (1).

LEMMA 3 ([4]). *Let $Y_1, Y_2, \dots, Y_k$ be 1-dependent random variables with $\mathrm{E}|Y_j|^2 < \infty$, $j = 1, \dots, k$. Then*

$$|\widehat{\mathrm{E}}Y_1Y_2\dots Y_j| \leqslant 2^{j-1}\prod_{k=1}^{j}\sqrt{\mathrm{E}|Y_k|^2}.$$

LEMMA 4. *Let condition* (1) *be satisfied. Then, for $k \geqslant 7$, we have*

$$\begin{aligned}\varphi_k - 1 = {} & A_1 + A_2 + A_3 - A_2A_1 + A_4 - 2A_3A_1 + A_2\big(A_1^2 - A_2\big)\\ & + A_5 - 3A_4A_1 + A_3\big(3A_1^2 - 3A_2\big) + 3A_2^2A_1\\ & + A_6 - 4A_5A_1 + A_3\big(12A_2A_1 - 2A_3\big) + A_2\big(2A_2^2 - 4A_4\big)\\ & + A_7 + A_3\big(10A_2^2 - 5A_4\big) - 5A_2A_5 + C_9\theta p^4|x|^4. \end{aligned} \tag{4}$$

Moreover, for $k = 2, 3, 4, 5, 6$ *the estimate* (4) *holds with* $A_3 = A_4 = A_5 = A_6 = A_7 = 0$, $A_4 = A_5 = A_6 = A_7 = 0$, $A_5 = A_6 = A_7 = 0$, $A_6 = A_7 = 0$, $A_7 = 0$, *respectively.*

Proof. Applying Lemma 1 we obtain

$$\varphi_k - 1 = A_1 + \frac{A_2}{\varphi_{k-1}} + \frac{A_3}{\varphi_{k-2}\varphi_{k-1}} + \frac{A_4}{\varphi_{k-3}\varphi_{k-2}\varphi_{k-1}} + \frac{A_5}{\varphi_{k-4}\varphi_{k-3}\varphi_{k-2}\varphi_{k-1}} + \frac{A_6}{\varphi_{k-5}\varphi_{k-4}\varphi_{k-3}\varphi_{k-2}\varphi_{k-1}} + \frac{A_7}{\varphi_{k-6}\varphi_{k-5}\varphi_{k-4}\varphi_{k-3}\varphi_{k-2}\varphi_{k-1}} + \sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(\mathrm{e}^{zX_j} - 1)(\mathrm{e}^{zX_{j+1}} - 1)\dots(\mathrm{e}^{zX_k} - 1)}{\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}}. \tag{5}$$

and

$$\frac{1}{|\varphi_k|} \leqslant \frac{1}{1 - |1 - \varphi_k|} \leqslant \frac{5}{4}.$$

From Lemma 3 we obtain

$$\big|\widehat{\mathrm{E}}\big(\mathrm{e}^{zX_j} - 1\big)\big(\mathrm{e}^{zX_{j+1}} - 1\big)\dots\big(\mathrm{e}^{zX_k} - 1\big)\big| \leqslant 2^{k-j} \prod_{m=j}^{k} \sqrt{\mathrm{E}\big|\mathrm{e}^{zX_m} - 1\big|^2} = 2^{k-j}|x|^{k-j+1} p^{(k-j+1)/2} \leqslant C_{10} p^4 |x|^4 (4\sqrt{p})^{k-j-7}.$$

Noting that, due to (1), we can assume p to be small. Therefore,

$$\sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(\mathrm{e}^{zX_j} - 1)(\mathrm{e}^{zX_{j+1}} - 1)\dots(\mathrm{e}^{zX_k} - 1)}{\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}} \leqslant C_{11} p^4 |x|^4 \sum_{j=1}^{k-7} (5\sqrt{p})^{k-j-7} \leqslant C_{12} p^4 |x|^4. \tag{6}$$

From (5) and (6) we get

$$\frac{1}{\varphi_k} = 1 + (1 - \varphi_k) + (1 - \varphi_k)^2 + C_{13}\theta p^3 |x|^4 = 1 - A_1 - \frac{A_2}{\varphi_{k-1}} - \frac{A_3}{\varphi_{k-1}\varphi_{k-2}} - \frac{A_4}{\varphi_{k-1}\varphi_{k-2}\varphi_{k-3}} - \frac{A_5 x^5}{\varphi_{k-1}\varphi_{k-2}\varphi_{k-3}\varphi_{k-4}} + (1 - \varphi_k)^2 + C_{14}\theta p^3 |x|^4$$
$$= 1 - A_1 + A_1^2 - A_2 + 3A_2A_1 - A_3 + 2A_2^2 + 4A_3A_1 - A_4 + 5A_2A_3 - A_5 + C_{15}\theta p^3 |x|^4.$$

Putting the last expression into (5) we complete the proof of Lemma 4, for $k \geqslant 7$. The cases $k = 2, 3, 4, 5, 6$ are proved similarly.

LEMMA 5. *Let condition* (1) *be satisfied. Then, for* $k \geqslant 7$,

$$\ln \varphi_k = A_1 + C_{16}\theta\big(R_1|x|^2 + R_2|x|^3 + R_3|x|^4\big).$$

Proof. Applying Lemma 1 and Lemma 4, it is not difficult to show that

$$\ln \varphi = A_1 + C_{17}\theta\bigg(\sum_{j=2}^{7}|A_j| + |A_1|^2 + p^2|x|^4\bigg).$$

Since $x = -\bar{x} - |x|^2$ one can easily obtain the following estimates

$$A_1 = (p_1 - p_{-1})x - p_{-1}|x|^2,$$
$$A_1^2 = C_{17}\theta\big(p|p_1 - p_{-1}||x|^2 + p^2|x|^4\big),$$
$$|A_2| \leqslant \big|a(-1,-1) - a(-1,1) - a(1,-1) + a(1,1)\big|\,|x|^2 + \big|a(-1,1) + a(1,-1) - 2a(1,1)\big|\,|x|^3 + \big|a(1,1)\big|\,|x|^4,$$
$$|A_3| \leqslant \sum_{j,k\in\{-1,1\}}\big|a(j,k,-1) - a(j,k,1)\big|\,|x|^3 + \sum_{j,k\in\{-1,1\}}\big|a(j,k,-1)\big|\,|x|^4 + |A_2|.$$

Let $a(j_1, j_2, \ldots, j_k) = P(X_1 = j_1, X_2 = j_2, \ldots, X_k = j_k) - P(X_1 = j_1)P(X_2 = j_2)\ldots P(X_k = j_k)$ and let $\sum_k^*$ denote the sum over all $j_1, \ldots, j_k \in \{-1, 1\}$. Then, for $k = 4, 5, 6, 7$,

$$|A_k| \leqslant \sum\nolimits_k^* \big|a(j_1, j_2, \ldots, j_k)\big|\,|x|^k + C_{18}\sum_{m=2}^{k-1}|A_m|.$$

Lemma's statement now follows from the following estimate

$$\big|a(j_1, j_2, \ldots, j_k)\big| \leqslant C_{19}\big|a(j_1, j_2)\big| + C_{20}p^2.$$

LEMMA 6. *Let* (1) *be satisfied and let* $|t| \leqslant \pi$. *Then*

$$\big|\widehat{F}_n(t) - \widehat{D}^n(t)\big| \leqslant C_{21}\exp\big\{-C_{22}npt^2\big\}\big[R_1|t|^2 + R_2|t|^3 + R_3|t|^4\big].$$

Proof. Applying Lemma 2 we get

$$\big|\widehat{F}_n(t) - \widehat{D}^n(t)\big| \leqslant C_{23}\exp\big\{-C_{22}npt^2\big\}\big|\ln\widehat{F}_n(t) - \ln\widehat{D}^n(t)\big|.$$

Note that $\ln\widehat{F}_n(t) = \sum_{k=1}^n \ln\varphi_k$. Consequently, from Lemma 5 we get the required estimate.

Theorems proof now follows from Lemma 6, Tsaregradskii's inequality

$$\sup_x\big|F_n\big\{(-\infty, x]\big\} - D^n\big\{(-\infty, x]\big\}\big| \leqslant \frac{1}{4}\int_{-\pi}^{\pi}\frac{\big|\widehat{F}_n(t) - \widehat{D}^n(t)\big|}{|t|}\,\mathrm{d}t$$

and formula of inversion for the local probability

$$\sup_x \left|F_n\{x\} - D^n\{x\}\right| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\widehat{F}_n(t) - \widehat{D}^n(t)\right| \mathrm{d}t.$$

References

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REZIUMĖ

J. Petrauskienė, V. Čekanavičius. 1-priklausomų indikatorių sumos Puasono tipo aproksimacija

1-priklausomų indikatorių suma aproksimuojama sudėtinio Puasono tipo skirstiniu. Gauti tolygūs Kolmogorovo ir lokalus aproksimacijos tikslumo įverčiai.

Raktiniai žodžiai: m-priklausomi atsitiktiniai dydžiai, sudetinis Puasono skirstinys, tolygi Kolmogorovo metrika, lokali metrika.