Liet. mat. rink. LMD darbai, **48/49**, 2008, 38–43

Note on arithmetical functions and multiples

Vilius STAKĖNAS (VU)
e-mail: vilius.stakenas@mif.vu.lt

Abstract. The existence of the logarithmic and number-theoretic densities of some sets related to arithmetical functions is investigated. The Dirichlet convolution is used for the representation of these functions.

Keywords: arithmetical functions, Dirichlet convolution, multiples.

The set of arithmetical functions $\mathcal{A} = \{f : f : \mathbb{N} \to \mathbb{R}\}$ with the Dirichlet convolution

$$(f * g)(n) = \sum_{d|n} f(n) g\Big(\frac{n}{d}\Big)$$

is a ring of functions with the unity element $e(n)$, where $e(1) = 1$ and $e(n) = 0$ if $n > 1$. We denote as usual by $\mu(n)$ the Möbius function, and by $\omega(n), \Omega(n)$ the numbers of primes dividing n counted without and with multiplicity. We use the concepts of additive and multiplicative functions in the usual number-theoretic sense.

An arbitrary arithmetical function f can be viewed as a result of convolution of some arithmetical function w and the constant function $I(n) = 1$:

$$f(n) = (w * I)(n) = \sum_{d|n} w(d), \quad w(d) = (f * \mu)(d).$$

We use this representation as generic and write $f(n) = f(n|w)$. It is our aim to investigate some relations between the conditions set on w and properties of $f(n|w)$.

It is easy to find out which functions $w(n)$ generate additive or multiplicative functions $f(n|w)$.

THEOREM 1. *The function $f(n|w)$ is multiplicative if and only if $w(n)$ is multiplicative.*

The function $f(n|w)$ is additive if and only if $w(n) = 0$ for all n with the condition $\omega(n) \neq 1$.

Proof. The first statement can be found in most textbooks of number theory.

Let us prove the second statement. It is obvious, that the conditions on w imply additivity of $f(n|w)$. We prove that these conditions are necessary. It can be done easily by induction over the values of $\Omega(n)$. Obviously, $f(1) = w(1) = 0$. If $\Omega(n) = \omega(n) = 2$, then $n = pq$, where p, q are both primes. If $f(n|w)$ is additive, then

$$f(n|w) = f(p|w) + f(q|w) = w(p) + w(q) = w(p) + w(q) + w(pq),$$

and $w(n) = 0$ follows. Let the statement be true for all n with the condition $2 \leqslant \omega(n) \leqslant \Omega(n) \leqslant m$. Let for some n, $\omega(n) \geqslant 2$, $\Omega(n) = m+1$. Then $n = n' p^a$, where p is prime and $(n', p) = 1$. We have

$$f(n|w) = f(n') + f(p^a) = \sum_{d'|n'} w(d') + \sum_{b \leqslant a} w(p^b) + \sum_{\substack{\delta | n', \delta > 1 \\ 1 \leqslant b \leqslant a}} w(\delta p^b).$$

The last sum is zero and for each δp^b, except for the largest $\delta p^b = n' p^a$, the condition $2 \leqslant \omega(\delta p^b) \leqslant \Omega(\delta p^b) \leqslant m$ is satisfied. Hence $w(\delta p^b) = 0$, and $w(n) = w(n' p^a) = 0$. The theorem is proved.

For an arbitrary subset of natural numbers $A \subset \mathbb{N}$ we denote the set of multiples

$$\mathcal{M}(A) = \bigcup_{a \in A} \{n \in \mathbb{N} \colon n \equiv 0 \ (\mathrm{mod}\ a)\}.$$

If $w(d) \in \{0, 1\}$ and $A_w = \{d \colon w(d) = 1\}$, then $f(n|w) > 0$ holds only if $n \in \mathcal{M}(A_w)$. The value of $f(n|w)$, if $f(n|w) > 0$, can be interpreted as the „weight" of the multiple n in the obvious sense.

We introduce two systems of densities. If $A \subset \mathbb{N}$ and $x > 1$, let us denote

$$\nu_x\{A\} = \frac{\#(A \cap (0, x])}{\lfloor x \rfloor}, \quad \lambda_x\{A\} = L^{-1} \sum_{\substack{n \in A \\ n \leqslant x}} \frac{1}{n}, \quad L = \sum_{n \leqslant x} \frac{1}{n}.$$

We denote the lower and the upper limits of $\nu_x\{A\}, \lambda_x\{A\}$, as $x \to \infty$, by $\underline{\nu}\{A\}, \overline{\nu}\{A\}$, $\underline{\lambda}\{A\}, \overline{\lambda}\{A\}$, respectively. It is well known that for all subsets $A \subset \mathbb{N}$

$$\underline{\nu}\{A\} \leqslant \underline{\lambda}\{A\} \leqslant \overline{\lambda}\{A\} \leqslant \overline{\nu}\{A\}.$$

If $\underline{\nu}\{A\} = \overline{\nu}\{A\}$, we denote this value by $\nu\{A\}$ and say that A possess the number-theoretic density. If $\underline{\lambda}\{A\} = \overline{\lambda}\{A\} = \lambda\{A\}$, we say that A has the logaritmic density.

We are going to prove some facts about the existence of densities for the sets $\{n \colon f(n|w) \geqslant z\}$.

THEOREM 2. *If the function $w(n)$ satisfies*

$$\sum_{w(d) \neq 0} \frac{1}{d} < \infty,$$

then for any z the density $\nu\{n \colon f(n|w) \geqslant z\}$ exists.

Proof. Let $d_1 < d_2 < \ldots$ be the sequence of all numbers with the property $w(d) \neq 0$. Let $\epsilon > 0$ and N be some number such that

$$\sum_{j > N} \frac{1}{d_j} \leqslant \epsilon.$$

Define a function w^* taking $w^*(d) = w(d)$, if $d = d_j$ with $j \leqslant N$, and $w^*(d) = 0$ otherwise. Then

$$\overline{\nu}\{n\colon f(n|w) \neq f(n|w^*)\} \leqslant \sum_{j>N} \frac{1}{d_j} \leqslant \epsilon,$$

and

$$\underline{\nu}\{f(n|w^*) \geqslant z\} - \epsilon \leqslant \underline{\nu}\{f(n|w) \geqslant z\} \leqslant \overline{\nu}\{f(n|w) \geqslant z\} \leqslant \overline{\nu}\{f(n|w^*) \geqslant z\} + \epsilon.$$

Hence, it suffices to show the existence of $\nu\{f(n|w^*) \geqslant z\}$.

Let $\mathbb{D} = \{d_1, d_2, \ldots, d_N\}$. For each non-empty subset $D \subset \mathbb{D}$ we denote by $m(D)$ the least common multiple of numbers in D. The numbers $m(D_1), m(D_2)$ indexed by different subsets D_1, D_2 are not necessarily different. We avoid repetitions in the following way: if a is some number in the sequence, find all numbers $m(D_j) = a$ and remove them, except the number indexed by $\cup D_j$, i. e. leave the number $a = m(\cup D_j)$. Let $\mathbb{M}$ be the set of all remaining (different) numbers with $M = m(\mathbb{D})$ the largest of them.

If $n \equiv m \pmod M$ with some $m \in \mathbb{M}$, then $f(n|w^*) = f(m|w^*)$; if $m \notin \mathbb{M}$, then $f(n|w^*) = 0$. Hence for all values a the densities $\nu\{f(n|w^*) = a\}$ exist, and this suffices for the proof.

We turn now to the question of existence $\lambda\{f(n|w) \geqslant z\}$. We use in our reasoning the fact established by Erdös and Davenport: for any subset $A \subset \mathbb{N}$ the set of multiples $\mathcal{M}(A)$ has the logarithmic density (see [3]; [4] Th. 12, p.258; [5] Th. 02, p.5).

THEOREM 3. *If the function $w(n)$ satisfies*

$$\sum_{w(d)<0} \frac{1}{d} < \infty,$$

then for any z the density $\lambda\{n\colon f(n|w) \geqslant z\}$ exists.

Proof. As in the proof of the previous theorem we reduce the proof to the case of function w with the finite number of d satisfying $w(d) < 0$. Let $\mathbb{D}$ be the set of all d such that $w(d) < 0$. We repeat all the arguments of the proof of the previous theorem leading from the set $\mathbb{D}$ to the set of different multiples $\mathbb{M}$ and $M = m(\mathbb{D})$.

Let $w_+(d) = \max\{w(d), 0\}$, $w_-(d) = \min\{w(d), 0\}$. Then

$$f(n|w) = f(n|w_-) + f(n|w_+).$$

The function $f(n|w_+)$ has a nice property: for every u

$$\mathcal{M}(\{n\colon f(n|w_+) \geqslant u\}) = \{n\colon f(n|w_+) \geqslant u\}.$$

Hence we get from the Erdös-Davenport result that $\lambda\{n\colon f(n|w_+) \geqslant u\}$ exists.

Note, that if $m \in \mathbb{M}$ and $n \equiv m \pmod M$, then

$$f(n|w) = f(n|w_-) + f(n|w_+) = f(m|w_-) + f(m|w_+), \quad f(m|w_-) < 0. \tag{1}$$

If $m \notin \mathbb{M}$, then (1) holds with $f(m|w_-) = 0$, too. This gives a chance to split the set $\{n\colon f(n|w) \geqslant z\}$ into disjunctive parts:

$$\{n\colon f(n|w) \geqslant z\} = \bigcup_{m=0}^{M-1} \{n\colon n \equiv m \pmod M),\ f(n|w_+) \geqslant z - f(m|w_-)\}.$$

We conclude the proof using the following helpful fact: if $A \subset \mathbb{N}$ and q, Q are some natural numbers, then the logarithmic density

$$\lambda\{n\colon n \equiv q \pmod Q),\ n \in \mathcal{M}(A)\}$$

exists. In the case $(q, Q) = 1$ it is proved in [5] (Lemma 1.17, p.61). To show, that it holds as $(q, Q) > 1$, is easy. Observe now that with $u = z - f(m|w_-)$ in the definition of the set

$$\{n\colon n \equiv m \pmod M),\ f(n|w_+) \geqslant u\},$$

the condition $f(n|w_+) \geqslant u$ can be replaced by $n \in \mathcal{M}(\{n\colon f(n|w_+) \geqslant u\})$; hence this set has the logarithmic density. The proof is complete.

Now we look for an example of function such that for $A_z = \{n\colon f(n|w) \geqslant z\}$ the density $\lambda\{A_z\}$ exists, but $\overline{\nu}\{A_z\} - \underline{\nu}\{A_z\} > 0$ for each z. In the construction of such function we use the following result of Erdös ([2]):

$$\nu\{\mathcal{M}([T; 2T))\} \to 0, \quad T \to \infty, \tag{2}$$

here $[T; 2T)$ means the set of natural numbers in this interval. The existence of densities in (2) can be proved using the combinatorial including-excluding principle, which works because of finitness of $[T; 2T)$.

Let $k \geqslant 1$ be some natural number. We have, obviously, that

$$\mathcal{M}([T; 2^k T)) = \bigcup_{j=0}^{k-1} \mathcal{M}([2^j T; 2^{j+1} T)).$$

It follows then from (2) that

$$\nu\{\mathcal{M}([T; 2^k T))\} \to 0, \quad T \to \infty. \tag{3}$$

THEOREM 4. *Let $c > 0$ and $0 < \delta < 1$ be some real numbers. There exists some function $f(n|w)$ such that for all $z \geqslant c$ the densities $\lambda\{n\colon f(n|w) \geqslant z\}$ exist, and $\overline{\nu}\{n\colon f(n|w) \geqslant z\} - \underline{\nu}\{n\colon f(n|w) \geqslant z\} \geqslant \delta$.*

Proof. Let k be some natural number such that $1 - 2^{-k} \geqslant (1+\delta)/2$ and $\epsilon = (1-\delta)/2$. According to (3) we can choose the sequence of natural numbers $T_m, T_{m+1} > 2^k T_m$ with the conditions

$$\sum_m \nu\{\mathcal{M}([T_m; 2^k T_m))\} < \epsilon/2,$$

$$\nu_x\{\mathcal{M}([T_m; 2^k T_m)\} < 2 \cdot \nu\{\mathcal{M}([T_m; 2^k T_m)\} \text{ as } x \geqslant T_{m+1}.$$

Let $z_1 = c, z_1 < z_2 < \ldots$ be an arbitrary unbounded sequence. We define a function $w(d)$ taking $w(d) = z_m$, if $d \in [T_m; 2^k T_m)$, and $w(d) = 0$, if $d \notin \cup_m [T_m; 2^k T_m)$. The existence of $\lambda\{f(n|w) \geqslant z\}$ follows from the previous theorem.

For fixed $z \geqslant c$ find some z_m such that $z \leqslant z_m$. Obviously,

$$\nu_x\{f(n|w) \geqslant z_m\} \leqslant \nu_x\{f(n|w) \geqslant z\} \leqslant \nu_x\{f(n|w) \geqslant c\}.$$

We show that $\overline{\nu}\{n\colon f(n|w) \geqslant z_m\} \geqslant 1 - 2^{-k}$ and $\underline{\nu}\{n\colon f(n|w) \geqslant c\} \leqslant \epsilon$. The second inequality follows from

$$\nu_{T_m}\{n\colon f(n|w) \geqslant c\} = \nu_{T_m}\{\cup_{j=1}^{m-1} \mathcal{M}([T_j, 2^k T_j))\} \leqslant 2 \sum_{j<m} \nu\{\mathcal{M}([T_j; 2^k T_j))\} < \epsilon.$$

We obtain the first one using the bound

$$\begin{aligned}\nu_{2^k T_{m+j}}\{n\colon f(n|w) \geqslant z_m\} &\geqslant \nu_{2^k T_{m+j}}\{\mathcal{M}([T_{m+j}; 2^k T_{m+j}))\} \\ &= \frac{2^k T_{m+j} - T_{m+j}}{2^k T_{m+j}} = 1 - 2^{-k}.\end{aligned}$$

This suffices for the proof.

We are now going to interpret the Behrend inequality for the set of multiples in the context of arithmetical functions. Let A, B be arbitrary subsets of natural numbers. The Behrend inequality is

$$1 - \lambda\{\mathcal{M}(A \cup B)\} \geqslant (1 - \lambda\{\mathcal{M}(A)\}) \cdot (1 - \lambda\{\mathcal{M}(B)\}), \tag{4}$$

see [1]; [5] Th. 012, p.15.

Let now $f(n|w_1), f(n|w_2)$ be two functions and $w_i(d) \geqslant 0$ for all d. With some fixed z_1, z_2 denote

$$A = \{n\colon f(n|w_1) \geqslant z_1\}, \quad B = \{n\colon f(n|w_2) \geqslant z_2\}.$$

Because of $\mathcal{M}(A) = A, \mathcal{M}(B) = B, \mathcal{M}(A \cup B) = A \cup B$, the sets possess the logaritmic densities. We have

$$\begin{aligned}1 - \lambda\{A\} &= \lambda\{n\colon f(n|w_1) < z_1\}, \\ 1 - \lambda\{B\} &= \lambda\{n\colon f(n|w_2) < z_2\}, \\ 1 - \lambda\{A \cup B\} &= \lambda\{n\colon f(n|w_1) < z_1, f(n|w_2) < z_2\}.\end{aligned}$$

Now from (4) we obtain

THEOREM 5. *If $w_1(d) \geqslant 0, w_2(d) \geqslant 0$, then for all z_1, z_2*

$$\lambda\{f(n|w_1) < z_1\} \cdot \lambda\{f(n|w_2) < z_2\} \leqslant \lambda\{f(n|w_1) < z_1, f(n|w_2) < z_2\}. \tag{5}$$

Evidently, (5) can be rewritten for more than two functions.

For which additive or multiplicative functions $f_1(n) = f(n|w_1)$, $f_2(n|w_2)$ inequality (5) holds? Having in mind Theorem 1, we derive quickly the sufficient condition: it suffices that for any prime p

$$0 \leqslant f_i(p) \leqslant f_i(p^2) \leqslant \dots, \quad i = 1, 2, \dots,$$

holds. If the sets $\{n\colon f_i(n) < z_i\}$, $\{n\colon f_1(n) < z_1,\ f_2(n) < z_2\}$ possess the number-theoretic densities they can be used in (5) instead of logaritmic ones. For example, let P_1, P_2 be some arbitrary subsets of prime numbers; define the additive functions

$$f_i(n) = \sum_{p \in P_i,\, p^\alpha || n} (\alpha - 1), \quad i = 1, 2.$$

Then for all z_1, z_2

$$\nu\{n\colon f_1(n) < z_1\} \cdot \nu\{n\colon f_2(n) < z_1\} \leqslant \nu\{n\colon f_1(n) < z_1,\ f_2(n) < z_2\}.$$

References

1. F.A. Behrend, Generalization of an inequality of Heilbronn and Rohrbach, *B. Amer. Math. soc.*, **54**, 681–684.
2. P. Erdös, Note on sequences of integers no one of which is divisible by any other, *J. London Math. Soc.*, **10**, 126–128 (1935).
3. H. Davenport, P. Erdös, On sequences of positive integers, *Acta Arith.*, **2**, 147–151 (1937).
4. H. Halberstam, K.F. Roth, *Sequences*, Oxford University Press (1966).
5. R.R. Hall, *Sets of Multiples*, Cambridge University Press (1996).

REZIUMĖ

V. Stakėnas. Pastaba apie aritmetines funkcijas ir kartotinius

Straipsnyje nagrinėjamas kartotinių aibių ir aritmetinių funkcijų ryšys. Įrodomi teiginiai apie aritmetinių funkcijų reikšmių asimptotinius dažnius.

Raktiniai žodžiai: aritmetinės funkcijos, kartotiniai.