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Modified L-functions

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Abstract. The sequence of generalized prime numbers $q_0 = 1,\ q_n = p_{n+1}^k - 1,\ n \in \mathbb{N}$, and the corresponding zeta-function $Z_k(s) = \prod_{p>2} \left(1 - (p^k - 1)^{-s}\right)^{-1},\ s = \sigma + it$, are analyzed. The analyticity of $Z_k(s)$ in the domain $\sigma > 0$, except for a simple pole $s = \frac{1}{k}$, is proved.

Keywords: analyticity, generalized number, L-function, residue, zeta-function.

The elements of the sequence $\mathcal{P} = \{1,\ q_1,\ q_2,\ \ldots\}$ with conditions

$$q_0 = 1 < q_1 \leqslant q_2 \leqslant \ldots, \quad q_n \to \infty\ (n \to \infty)$$

are called generalized primes (g-primes). The multiplicative semigroup $1 = n_1 < n_2 \leqslant n_3 \leqslant \ldots$, generated by g-primes, is the system of generalized integers (g-integers). Such number systems were analyzed by Beurling [1].

The analytic method which uses the Riemann zeta-function and Dirichlet L-functions, is powerful in the research of distribution of rational primes. The analytic method in the theory of generalized number systems is successfully applied, too. In this case, we need to know analytic properties of modified zeta and L-functions attached to given systems of generalized numbers. We recall that above zeta and L-functions are defined, respectively, by

$$\zeta_{\mathcal{P}}(s) = \prod_{k=1}^{\infty} (1 - q_k^{-s})^{-1},$$

$$L_{\mathcal{P}}(s, \chi) = \prod_{k=1}^{\infty} (1 - \chi(q_k) q_k^{-s})^{-1}, \quad s = \sigma + it.$$

Various generalized number systems and corresponding zeta and L-functions were considered in [2]–[7]. For example, in [3] the analyticity of the function

$$\zeta(s, v) = \prod_p \left(1 - (vp)^{-s}\right)^{-1}, \quad s = \sigma + it,\ \sigma > 1,\ v > \frac{1}{2},$$

in the half-plane $\Re s > 0$ cut at zeros of powers of Riemann zeta-function is proved, and this is sufficient to obtain some results on the asymptotic distribution of g-integers.

The theory of generalized number systems may be applied for investigation of the values distribution of some classical arithmetic functions. For example, the Euler totient function $\varphi(n)$ is connected with the sequence of g-primes defined by "shifted" rational primes, i. e.,

$$q_0 = 1,\ q_n = p_{n+1} - 1, \quad n \in \mathbb{N}.$$

Analytic properties of the corresponding zeta-function

$$Z(s) = \prod_{p>2} \left(1 - (p-1)^{-s}\right)^{-1}$$

were discussed in [4], [7]. In the more general cases, zeta and L-functions

$$\zeta(s, v, r) = \prod_{p>m(r)} \left(1 - (v(p+r))^{-s}\right)^{-1}, \quad \sigma > 1,\ v > \frac{1}{2},\ r \in \mathbb{R}, \tag{1}$$

and

$$L(s, v, \chi) = \prod_{p} \left(1 - \chi(vp)(vp)^{-s}\right)^{-1}, \quad \sigma > 1,\ v > 1,\ v \in \mathbb{Z},$$

were considered in [2, 5, 7]. Here $m(r)$ is chosen so that all multipliers in (1) should be correctly defined.

In this paper, we consider the sequence of g-primes

$$q_0 = 1,\ q_n = p_{n+1}^k - 1, \quad n \in \mathbb{N},\ k \in \mathbb{N}, \tag{2}$$

and derive some analytic properties of the corresponding zeta-function

$$Z_k(s) = \prod_{p>2} \left(1 - (p^k - 1)^{-s}\right)^{-1}.$$

The sequence (2) of g-primes is associated with the arithmetic Jordan function

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right), \quad k \in \mathbb{N},$$

which is a generalization of the Euler function, namely, $J_1(n) = \varphi(n)$.

THEOREM. *The function $Z_k(s)$ is analytic in the domain $\sigma > 0$, except for a simple pole $s = \frac{1}{k}$ with residue*

$$R_k = \frac{1}{2k} \prod_{p>2} \left(\frac{1 - p^{-1}}{1 - (p^k - 1)^{-\frac{1}{k}}}\right). \tag{3}$$

Proof. We write the function $Z_k(s)$ in form

$$Z_k(s) = (1 - 2^{-ks})\zeta(ks)Q_k(s), \tag{4}$$

where $\zeta(s)$ is Riemann zeta-function, and

$$Q_k(s) = \prod_{p>2}\left(1 + \frac{(p^k-1)^{-s} - p^{-ks}}{1-(p^k-1)^{-s}}\right). \tag{5}$$

For $\sigma \geqslant \sigma_0 > 0$, uniformly with respect to k, we have that

$$|1-(p^k-1)^{-s}| \geqslant C(\sigma_0).$$

The numerator of the fraction in (5) is evaluated as follows:

$$|(p^k-1)^{-s} - p^{-ks}| = |s\int_{p^k-1}^{p^k} v^{-s-1}\,\mathrm{d}v| \leqslant \frac{|s|}{(p^k-1)^{1+\sigma}},$$

and this implies the analyticity of $Q_k(s)$ in the region $\sigma > 0$. A simple application of formula (4) gives the expression (3) for R_k, and this completes the proof of the theorem.

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REZIUMĖ

E. Stankus. Modifikuotosios L-funkcijos

Darbe įrodytas funkcijos

$$Z_k(s) = \prod_{p>2} \left(1 - (p^k - 1)^{-s}\right)^{-1},$$

susijusios su Žordano aritmetine funkcija

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right), \quad k \in \mathbb{N},$$

analiziškumas srityje $\sigma > 0$, išskyrus paprastą polių taške $s = \frac{1}{k}$ su rezidiumu

$$R_k = \frac{1}{2k} \prod_{p>2} \left(\frac{1 - p^{-1}}{1 - (p^k - 1)^{-\frac{1}{k}}}\right).$$

Įrodant šį teiginį funkcija $Z_k(s)$ užrašoma pavidalu

$$Z_k(s) = (1 - 2^{-ks})\zeta(ks) Q_k(s);$$

čia $\zeta(s)$ yra Rymano dzeta funkcija, o funkcija $Q_k(s)$ – analizinė pusplokštumėje $\sigma > 0$.

Raktiniai žodžiai: analiziškumas, apibendrintieji skaičiai, dzeta funkcija, L-funkcija, reziduumas.