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Information measures for the stochastic Gompertz growth model

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Abstract. In this paper we investigate the Shannon's, Fisher's and Tsallis' information measures for the Gompertz type stochastic process. We study the information measures of the stationary and non-stationary Gompertz type densities associated to the three parameters: intrinsic growth rate, saturation measure and noise amplitude. Finally we simulate the confidence interval for the value of all information measures. The results are implemented in the symbolic algebra language MAPLE.

Keywords: stochastic process, Gompertz, Shannon, Fisher, Tsallis, simulation.

Introduction

Noise driven dynamical systems is found in every branch of natural sciences. Stochastic models with additive or multiplicative noise find numerous applications in forestry [5], [6], biology [7], economy [1], [10]. The Shannon's information entropy [8], Fisher's information measure [2], Tsalis' information measure [9] and related quantities [4] are an appropriate tool for the study of non-stationary and stationary states in stochastic processes. The rate of change in time of information measures considers the phase space expansion of the stochastic process. It is observed that the Shannon entropy can grow, decay or show up a mixed behavior.

In this paper we suppose that dynamics of stochastic process of tree diameter growth is expressed in terms of the Gompertz type stochastic ordinary differential equation. The Gompertz stochastic growth law with multiplicative noise we use in growth modeling due to simple and attractive interpretation and admit close-form density solution. The close-form probability density function of tree diameter size facilitates explicit calculations of various information measures. We suppose that dynamics of tree diameter growth is expressed in terms of the stochastic ordinary differential equation with multiplicative noise in the following form [5]

$$\mathrm{d}X(t) = rX(t)\ln\frac{K}{X(t)}\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W(t), \quad t \in [0; T], \tag{1}$$

where t is the age of a forest stand, r is the diameter intrinsic growth rate, K is the diameter carrying capacity and forms a numerical upper bound on the diameter size, and $X(t)$ is the breast height diameter at the age t, σ is the intensity of noise, $W(t)$ is the standard Brownian motion (white noise). This diameter growth model, expressed as the ordinary stochastic differential Eq. (1), holds the transition probability density function $p(x,t)$, which project the distribution of tree diameter size subject to the age t.

Methods and results

Each solution of the stochastic differential Eq. (1) describes one path of evolution of process. The ensemble of realizations satisfies the corresponding Fokker–Plank equation

$$\frac{\partial p(x,t)}{\partial t} = -r\frac{\partial}{\partial x}\Big(x \ln \frac{K}{x} p(x,t)\Big) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\big(x^2 p(x,t)\big), \tag{2}$$

where $p(x,t)$ is the transition probability density function of the process $X(t)$.

The spreading of the transition probability density function $p(x,t)$ is best measured by the Shannon information entropy, also named differential entropy

$$S(t) = -\int p(x,t) \ln p(x,t)\,\mathrm{d}t. \tag{3}$$

The Shannon entropy is a well-known method for estimating the degree of disorder in dynamical system. The original definition of Shannon entropy conveys uncertainty and information measure. The less is the uncertainty of the system the large is the information that we acquire.

The Fisher information measure of $p(x,t)$ is given by

$$I(t) = \int \frac{1}{p(x,t)}\Big(\frac{\partial}{\partial x} p(x,t)\Big)^2 \mathrm{d}t, \tag{4}$$

which measures its sharpness or concentration. The Fisher information measure is always positive and reflects the localization characteristics of the probability distribution more sensitively than the Shannon measure.

The Tsallis entropy, defined by

$$S_q(t) = \frac{1}{q-1}\bigg(1 - \int \big(p(x,t)\big)^q \,\mathrm{d}t\bigg), \tag{5}$$

is an extension of Shannon entropy with one-real-parameter of q. In the limit of $q \to 1$, the Tsallis entropy (Eq. (5)) reduces to the Shannon entropy (Eq. (3)), since $(p(x,t))^{q-1} = \mathrm{e}^{(q-1)\ln p(x,t)} \approx 1 + (q-1)p(x,t)$.

Using transformation $Y(t) = \ln(X(t))$ and Ito's formula, we transform the nonlinear process (1) into the Ornstein-Uhlenbeck process [3]. So we can find the time dependent solution of the Fokker–Planck equation (2). The solution of Eq. (2) has the following form

$$p(x,t) = \frac{1}{\sigma x\sqrt{\pi(1-\mathrm{e}^{-2rt})/r}}\mathrm{e}^{-\frac{r(\ln x - \ln K + \sigma^2/2r - \mathrm{e}^{-rt}\ln x_0)^2}{\sigma^2(1-\mathrm{e}^{-2rt})}}. \tag{6}$$

The exact steady state solution $p(x)$ of Eq. (2) has the following form

$$p(x) = \frac{K}{\sigma}\sqrt{\frac{r}{\pi}}\mathrm{e}^{-\frac{\sigma^2}{4r}} x^{-2} \mathrm{e}^{-\frac{r\ln^2\frac{K}{x}}{\sigma^2}}. \tag{7}$$

These probability density functions are properly normalized and their moments, defined as $m_n(t) = M(X^n(t)) = \int x^n p(x,t)\,\mathrm{d}x$ (non-steady state), $m_n = M(X^n(t)) = \int x^n p(x)\,\mathrm{d}x$ (steady state), contain valuable information about the stochastic dynamics of tree diameter. The mathematical reasons for the usage of stochastic model we motivate by the distinction between the stochastic logistic growth model and its deterministic counterpart. For the stochastic Gompertz model of diameter growth we can derive the first two moments, namely, the mean and variance of tree diameter size. These equations showed that the mean diameter size of forest stands is extremely sensitive with respect to the form and size of the coefficient of volatility (amplitude of noise), and the deterministic Gomperz model overestimates the true mean diameter size in the presence of stochastic perturbations [6].

The importance of a stable steady state, as a criteria for biological well-being, is emphasized by many researches. Next we discuss a comparison of the non-stationary transition probability density function (6) and the stationary probability density (7). How far from each other these two probability densities are?

Now we illustrate these results by characterizing the underlying diameter dynamics as a stochastic process with the multiplicative noise. For model estimation were used observations of 1581 pines. The proposed information measures are applied to the forest data taken from the pine trees growing in different areas of Lithuania. The data source is based on the data provided by Lithuanian National Forest Inventory. The estimation of the parameters r, K, σ for the Gompertz stochastic logistic growth law (1) were calculated using the maximum likelihood procedure and the real data set from the observations of 1581 pine trees [5]. The estimates of parameters are: $r = 0.0647$, $K = 33.6718$, $\sigma = 0.1350$. For a given estimates of parameters and the stationary probability density (7), the information measures (3)–(5) take values: $S = 3.814597$, $F = 0.012550$, $S_2 = 0.973379$.

As we can see Fig. 1, the Tsallis entropy decreases monotonically as power q increases and reaches the Shannon entropy ($q \to 1$). For a given estimates of parameters and the transition probability density (6), information measures (3)–(5) are shown in Fig. 2.

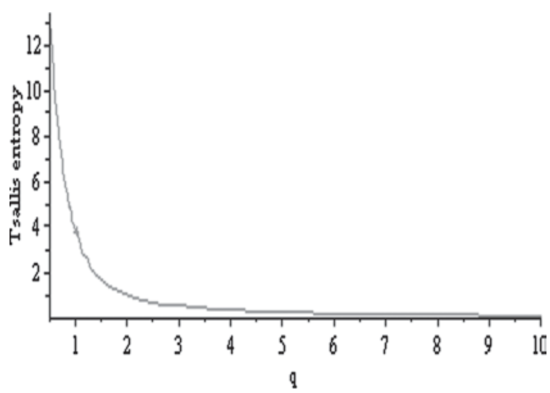

Fig. 1. Plot of the Tsallis entropy vs. power q.

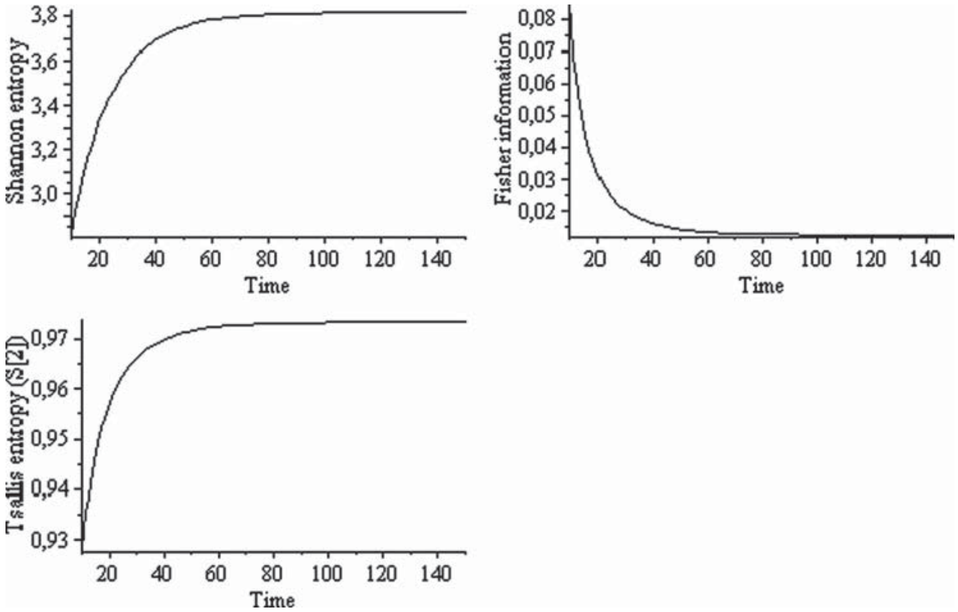

Fig. 2. Plot of differential information measures vs. time t.

A stable steady state in ecology is important for an understanding of ecosystems as dynamical complex processes. A departure from steady state indicates a negative unhealthy situation of ecosystem. The importance of stable steady state, as a criteria for biological well-being, emphasize many researches. This leads to the idea of measuring departure from steady state. In this paper the approach of obtaining a departure from steady state consists in defining the confidence intervals of steady state information measures using Monte Carlo simulations. The distances $S - S(t)$, $I - I(t)$, $S_2 - S_2(t)$ describe the loss of information when the non-steady state probability density $p(x, t)$ is used to approximate the steady state probability density $p(x)$, and measure the difference between the tree diameter maximum entropy and its entropy at given age t. While smaller these distances clearly indicate less information is lost by the non-steady state probability density function, there is no absolute scale against which to judge the significance of distances $S - S(t)$, $I - I(t)$, $S_2 - S_2(t)$. So it is worth noting that significance of these distances might be redefined through one-sided confidence intervals, at first providing lower (upper) bound for the steady state Shannon, Tsallis, Kullback (Fisher) information measure, and then by comparing values of the steady state lower (upper) bound and the non-steady state curve.

We carried out a small simulation study to compare the non-steady state time evolution of information measures with its steady state confidence interval. We repeatedly ($s = 1000$) simulated 1581 points distributed by probability density function (7) ($r = 0.0647$, $K = 33.6718$, $\sigma = 0.1350$). Then repeatedly the parameters r, K, σ were estimated by maximum likelihood procedure and the information measures (3)–(5)

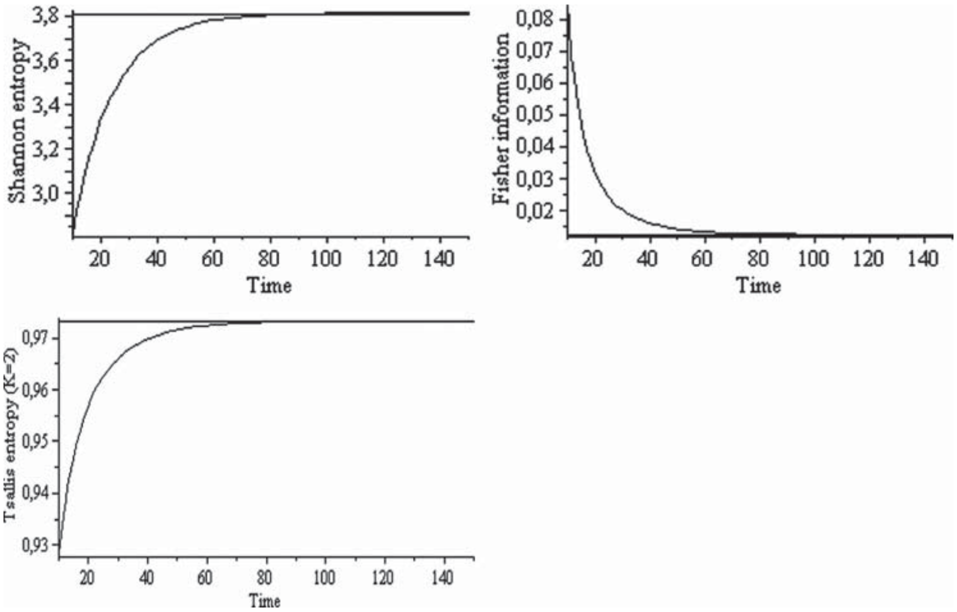

Fig. 3. Plot of information measures vs. time t and corresponding steady state values.

were calculated. The 95% lower bound for the steady state Shannon and Tsallis entropy are 3.8091, 0.9635 and upper bound for the steady state Fisher information measure is 0.0202. The hitting times of time dependent information measures to its stationary confidence bound values are presented in Fig. 3. The Shannon differential entropy $S(t)$ hits a 95% lower bound at 88 year, the Fisher information $I(t)$ hits a 95% upper bound at 93 year, and the Tsallis entropy $S_2(t)$ hits a 95% lower bound at 85 year. All aforementional information measures are closely related. The comparison of the results of optimal harvesting periods for all used information measures are presented in Fig. 3. As we see in Fig. 3, the optimal harvesting defined by Shannon's entropy has minimal periods and optimal harvesting defined by Fisher's information has maximal periods. Most forest stands reach their economically optimal harvesting period prior to our defined biologically optimal harvesting period.

Conclusions

In this paper we studied the steady state and non-steady state properties of a tree diameter growth model in the presence of multiplicative noise. We suggested a new approach in investigating the dynamics of diameter growth data, on the basis of the information measures (Shannon, Fisher, Tsallis). This viewpoint is completely novel in forest research projects. To our knowledge no studies had been performed on diameter data using such nonlinear methodologies. Therefore, the results presented in this paper encourage the use of these techniques for analyzing the optimal rotation period of forest stands. Other growth models such as Verhulst, Mitscherlich, von Bertalanffy, Richards, and much more, are equally plausible, but they lead to a numerical solution of transition probability density.

References

1. L.H.R. Alvarez, E. Koskela, Wicksellian theory of forest rotation under interest rate variability, *J. Econ. Dynam. Control*, **29**, 529–545 (2005).
2. R.A Fisher, On the mathematical foundations of theoretical statistics, *Prilos., Trans., R., Soc. Lond.*, Ser. A, **222**, 309–368 (1922).
3. I.I. Gihman, A.V. Skorohod, *Stochastic Differential Equations*, Springer-Verlag, Berlin (1972).
4. P. Gorbaczewski, Differential entropy and time, *Entropy*, **7**(4), 253–299 (2005).
5. P. Rupšys, E. Petrauskas, J. Mažeika, R. Deltuvas, The Gompertz type stochastic growth law and a tree diameter distribution, *Baltic Forestry*, **13**, 197–206 (2007).
6. P. Rupšys, The relationships between the diameter growth and distribution laws, *WSEAS Transactions on Biology and Biomedicine*, **4**(11), 15–21 (2007).
7. P. Rupšys, Delayed stochastic logistic growth laws in single-species population growth modeling, in: P. Anninos, T. Pham, and A. Grebennikov (Eds.) *Advanced Topics on Mathematical Biology and Ecology*, Mexico (2008), pp. 29–34.
8. C. Shannon, A mathematical theory of communication, *Bell System Tech. J.*, **27**, 379–423, 623–656 (1948).
9. C. Tsallis, Possible generalization of Boltzmann–Gibbs statistics, *J. Stat. Phys.*, **52**, 479 (1988).
10. Y. Willassen, The stochastic rotation problem: a generalization of Faustman's formula to stochastic forest growth, *J. Econ. Dynam. Control*, **22**, 573–596 (1998).

REZIUMĖ

P. Rupšys. Stochastinio Gompertco tipo augimo modelio informaciniai matai

Darbe augimo procesui modeliuoti yra naudojamas stochastinis Gompertco tipo modelis. Augimo proceso dinamikos charakterizavimui panaudojami Šenono, Fišerio, Tsalio informaciniai matai. Rezultatai iliustruojami panaudojant Lietuvos pušies medynų stebėjimo duomenis.

Raktiniai žodžiai: stochastinis procesas, Gompertcas, Fišeris, Tsalis, imitavimas.