Liet. mat. rink. LMD darbai, **48/49**, 2008, 72–77

On the Green's formula for a Stokes type problem

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Abstract. A time-periodic Stokes problem is studied in the domain with cylindrical outlets to infinity. Using the Fourier series the problem is reduced to a sequence of elliptic problem. For each of these elliptic boundary value problems a generalized Green's formula is constructed. The analogous Green's formula for the steady Stokes problem was obtained in [1].

Keywords: cylindrical outlets to infinity, time-periodic Stokes problem, generalized Green's formula.

1. Formulation of the problem

Let $\Omega \subset \mathbb{R}^3$ be a domain with cylindrical outlets to infinity, i.e., outside the ball $B_R = \{x \in \mathbb{R}^3\colon |x| \leqslant R\}$ the domain Ω coincides with a system of J semi-infinite cylinders Π_+^j of a constant cross section ω^j. Let $\Pi_+^j \cap \Pi_+^k = \emptyset$, $j \neq k$ and let the boundary $\partial\Omega$ be smooth. We consider in Ω the time-periodic Stokes problem

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \quad (x,t) \in \Omega \times (0, 2\pi), \tag{1}$$

$$-\nabla \cdot \mathbf{v} = \mathbf{0}, \quad (x,t) \in \Omega \times (0, 2\pi), \tag{2}$$

$$\mathbf{v} = \mathbf{0}, \quad (x,t) \in \partial\Omega \times (0, 2\pi), \tag{3}$$

$$\mathbf{v}(x,0) = \mathbf{v}(x, 2\pi), \quad x \in \Omega. \tag{4}$$

We assume that the external force $\mathbf{f} = (f_1, f_2, f_3)^t$ is 2π-time-periodic function. Problem (1)–(4) could be decomposed into a sequence of elliptic problems. Indeed, we can look for the solution to problem (1)–(4) in the form

$$\mathbf{v}(x,t) = \frac{\mathbf{v}_{c0}}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\{\mathbf{v}_{ck}(x)\cos kt + \mathbf{v}_{sk}(x)\sin kt\}, \tag{5}$$

$$p(x,t) = \frac{p_{c0}}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\{p_{ck}(x)\cos kt + p_{sk}(x)\sin kt\}. \tag{6}$$

*This work was partly supported by the Lithuanian Science Council Student Research Fellowship Award.

Inserting series (5), (6) into equations and boundary conditions we get for coefficients $\mathbf{v}_{ck}, \mathbf{v}_{sk}, p_{ck}, p_{sk}$ series of the systems of elliptic problems

$$\begin{cases} k\mathbf{v}_{sk} - \nu\Delta\mathbf{v}_{ck} + \nabla p_{ck} = \mathbf{f}_{ck}, & x \in \Omega, \\ -k\mathbf{v}_{ck} - \nu\Delta\mathbf{v}_{sk} + \nabla p_{sk} = \mathbf{f}_{sk}, & x \in \Omega, \\ -\nabla\cdot\mathbf{v}_{ck} = 0,\ -\nabla\cdot\mathbf{v}_{sk} = 0, & x \in \Omega, \\ \mathbf{v}_{ck} = \mathbf{0},\ \mathbf{v}_{sk} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \tag{7}$$

Here $\mathbf{f}_{c0}/(2\pi), \mathbf{f}_{ck}/\pi, \mathbf{f}_{sk}/\pi, k = 0, 2, \ldots,$ are Fourier coefficients of the function $\mathbf{f} = \mathbf{f}(x,t)$.

In this paper we derive so-called generalized Green's formula for problem (7). The analogous Green's formula for the steady Stokes problem was obtained in [1]. The obtained below results are important for the construction of correct asymptotic conditions at infinity which describe real time-periodic physical processes (for example bloodstream).

2. The asymptotics of the solution to problem (7)

Let $x^j = (x_1^j, x_2^j, x_3^j)$ be the local coordinate system related to the cylinder Π_+^j such that the axis x_3^j is directed along cylinder axis. We consider problem (7) in a weighted Sobolev space $W_\beta^l(\Omega)$ which is a closure of $C_0^\infty(\overline{\Omega})$ (a class of infinitely differentiable functions with compact supports in $\overline{\Omega}$) with respect to the norm

$$\left\|u;\, W_\beta^l(\Omega)\right\|^2 = \sum_{|\alpha|\leqslant l} \int_\Omega \rho_\beta(x)\left|D_x^\alpha u(x)\right|^2 \mathrm{d}x,$$

where ρ_β is a smooth positive function on $\overline{\Omega}$ such that $\rho_\beta(x) = \exp(\beta x_3^j)$ on $\Pi_+^j \setminus B_R,\ j = 1, \ldots, J$. If $\beta > 0$, elements of this space exponentially vanish as x_3^j tends to infinity, and they may exponentially grow, if $\beta < 0$.

Consider problem (7) in the cylinder Π_+^j. Using the methods of the book [2] and arguing in the same way as in [1] we obtain four special solutions of the homogeneous problem (7):

$$\mathbf{u}_{ck}^{j0} = (0,0,0,1,0,0,0,0)^t, \quad \mathbf{u}_{ck}^{j1} = (0,0,\varphi_k^j, x_3^j, 0, 0, -\psi_k^j, 0)^t, \tag{8}$$

$$\mathbf{u}_{sk}^{j0} = (0,0,0,0,0,0,0,1)^t, \quad \mathbf{u}_{sk}^{j1} = (0,0,\psi_k^j, 0, 0, 0, \varphi_k^j, x_3^j)^t, \tag{9}$$

where the pair of functions (φ_k^j, ψ_k^j) is the unique solution of the problem

$$\begin{cases} k\psi_k^j + \nu\Delta\varphi_k^j = 1, & x^{j'} = (x_1^j, x_2^j) \in \omega^j, \\ k\varphi_k^j - \nu\Delta\psi_k^j = 0, & x^{j'} \in \omega^j, \\ \varphi_k^j = \psi_k^j = 0, & x^{j'} \in \partial\omega^j. \end{cases} \tag{10}$$

According to Theorem 3.1.4 in [2] the sum of linear combinations of these solutions gives the main term (up to an exponentially vanishing term) of the asymptotic decomposition of the "growing" at infinity solution. Let $\chi_j(x)$ be a smooth cut-off function such that $\mathrm{supp}(\chi_j) \subseteq \Pi_+^j$ and $\chi_j(x) = 1$ if $x_3^j > L$ for $j = 1, \ldots, J$.

THEOREM 1. *Let $\beta > 0$. If $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \mathcal{D}_{-\beta}^l W(\Omega)$ is the solution to problem* (7) *with the right-hand side $\mathbf{f}_k = (\mathbf{f}_{ck}, \mathbf{f}_{sk}) \in W_\beta^{l-1}(\Omega)^6$, then*

$$\mathbf{u}_k(x) = \sum_{j=1}^{J} \chi_j(x) \left\{ a_{ck}^j \mathbf{u}_{ck}^{j0} + a_{sk}^j \mathbf{u}_{sk}^{j0} + b_{ck}^j \mathbf{u}_{ck}^{j1} + b_{sk}^j \mathbf{u}_{sk}^{j0} \right\} + \tilde{\mathbf{u}}_k, \tag{11}$$

where $\tilde{\mathbf{u}}_k \in \mathcal{D}_\beta^l W(\Omega)$, $a_{ck}^j, a_{sk}^j, b_{ck}^j, b_{sk}^j \in \mathbb{C}$. Here $\mathcal{D}_\beta^l W(\Omega) = W_\beta^{l+1}(\Omega)^6 \times W_\beta^l(\Omega)^2$.

3. Generalized Green's formula

Let $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}), \mathbf{U}_k = (\mathbf{V}_{ck}, P_{ck}, \mathbf{V}_{sk}, P_{sk}) \in C_0^\infty(\overline{\Omega})$. Integrating twice by parts in Ω one gets the standard Green's formula (see [3])

$$\begin{aligned}
&(-\nu\Delta\mathbf{v}_{ck} + \nabla p_{ck} + k\mathbf{v}_{sk}, \mathbf{V}_{ck})_\Omega + (-\nabla\cdot\mathbf{v}_{ck}, P_{ck})_\Omega \\
&\quad +(-\nu\Delta\mathbf{v}_{sk} + \nabla p_{sk} - k\mathbf{v}_{ck}, \mathbf{V}_{sk})_\Omega + (-\nabla\cdot\mathbf{v}_{sk}, P_{sk})_\Omega \\
&\quad +(\mathbf{v}_{ck}, \mathbf{n}P_{ck} - \nu\partial_\mathbf{n}\mathbf{V}_{ck})_{\partial\Omega} + (\mathbf{v}_{sk}, \mathbf{n}P_{sk} - \nu\partial_\mathbf{n}\mathbf{V}_{sk})_{\partial\Omega} \\
&\quad -(\mathbf{v}_{ck}, -\nu\Delta\mathbf{V}_{ck} + \nabla P_{ck} - k\mathbf{V}_{sk})_\Omega - (p_{ck}, -\nabla\cdot\mathbf{V}_{ck})_\Omega \\
&\quad -(\mathbf{v}_{sk}, -\nu\Delta\mathbf{V}_{sk} + \nabla P_{sk} + k\mathbf{V}_{ck})_\Omega - (p_{sk}, -\nabla\cdot\mathbf{V}_{sk})_\Omega \\
&\quad -(\mathbf{n}p_{ck} - \nu\partial_\mathbf{n}\mathbf{v}_{ck}, \mathbf{V}_{ck})_{\partial\Omega} - (\mathbf{n}p_{sk} - \nu\partial_\mathbf{n}\mathbf{v}_{sk}, \mathbf{V}_{sk})_{\partial\Omega} = 0,
\end{aligned} \tag{12}$$

here $(\,,)_\Omega$ stands for a scalar product in $L_2(\Omega)$. Denoting by $q(\mathbf{u}_k, \mathbf{U}_k)$ the left–hand side of the above formula we get

$$q(\mathbf{u}, \mathbf{U}) = q(\mathbf{U}, \mathbf{u}) = 0$$

for any $\mathbf{u} \in \mathcal{D}_\beta^l W(\Omega)$ and $\mathbf{U} \in \mathcal{D}_{-\beta}^l W(\Omega)$. Let S be an operator of problem (7) and S^* be an operator of the problem

$$\begin{cases}
-k\mathbf{V}_{sk} - \nu\Delta\mathbf{V}_{ck} + \nabla P_{ck} = \mathbf{F}_{ck}, & x \in \Omega, \\
k\mathbf{V}_{ck} - \nu\Delta\mathbf{V}_{sk} + \nabla P_{sk} = \mathbf{F}_{sk}, & x \in \Omega, \\
-\nabla\cdot\mathbf{V}_{ck} = 0,\ -\nabla\cdot\mathbf{V}_s k = 0, & x \in \Omega, \\
\mathbf{V}_{ck} = \mathbf{0},\ \mathbf{V}_{sk} = \mathbf{0}, & x \in \partial\Omega.
\end{cases} \tag{13}$$

It is clear that S^* is an adjoint operator to S with respect to the Green's formula (12). Note that S is not self-adjoint operator. Homogeneous problem (13) in the cylinder Π_+^j has four special solutions

$$\mathbf{U}_{ck}^{j0} = (0,0,0,1,0,0,0,0)^t, \qquad \mathbf{U}_{ck}^{j1} = (0,0,\varphi_k^j, x_3^j, 0,0,\psi_k^j, 0)^t, \tag{14}$$

$$\mathbf{U}_{sk}^{j0} = (0,0,0,0,0,0,0,1)^t, \quad \mathbf{U}_{sk}^{j1} = (0,0,-\psi_k^j, 0,0,0,\varphi_k^j, x_3^j)^t, \tag{15}$$

where functions φ_k^j and ψ_k^j are defined by formula (10). We denote by $\mathbb{D}_{\pm\beta}^l W(\Omega)$ the subset of functions $\mathbf{u}_k \in \mathcal{D}_{-\beta}^l W(\Omega)$ having expansion (11) and by $\mathbb{D}_{\pm\beta}^l W(\Omega)^*$ the subset of $\mathcal{D}^l_{-\beta}W(\Omega)$ consisting of functions having an expansions

$$\mathbf{U}_k = \sum_{j=1}^{J} \chi_j \left\{ A^j_{ck}\mathbf{U}^{j0}_{ck} + A^j_{sk}\mathbf{U}^{j0}_{sk} + B^j_{ck}\mathbf{U}^{j1}_{ck} + B^j_{sk}\mathbf{U}^{j1}_{sk} \right\} + \tilde{\mathbf{U}}_k, \tag{16}$$

where $\mathbf{U}^{jh}_{\diamond,k}, h \in \{0,1\}, \diamond \in \{c,s\}$, are defined by (14) and (15), $\tilde{\mathbf{U}}_k \in \mathcal{D}^l_{\beta}W(\Omega)$, $A^j_{ck}, A^j_{sk}, B^j_{ck}, B^j_{sk} \in \mathbb{C}$.
Since $\operatorname{supp}(\chi_j) \cap \operatorname{supp}(\chi_l) = \emptyset,\ j \neq l$, we have

$$q(\chi_j \mathbf{u}^{jh}_{\diamond,k}, \chi_l \mathbf{U}^{lm}_{\diamond\diamond,k}) = 0, \quad h,m \in \{0,1\}, \quad \diamond, \diamond\diamond \in \{c,s\}.$$

Using the fact that functions (8), (9) and (14), (15) are exact solutions to homogeneous problems (7) and (13), respectively, we get, after cumbersome computation, that

$$q(\chi_j \mathbf{u}^{jh}_{\diamond,k}, \chi_j \mathbf{U}^{jh}_{\diamond\diamond,k}) = 0, \quad h = 0,1, \quad \diamond, \diamond\diamond \in \{c,s\}.$$

Inserting representations (11) and (16) into $q(\mathbf{u}_k, \mathbf{U}_k)$ we get that a number of terms in $q(\mathbf{u}_k, \mathbf{U}_k)$ vanishes and, finally, we find

$$\begin{aligned} q(\mathbf{u}_k, \mathbf{U}_k) = \sum_{j=1}^{J} \Big\{ & a^j_{ck}\overline{B}^j_{ck}\, q(\chi_j \mathbf{u}^{j0}_{ck}, \chi_j \mathbf{U}^{j1}_{ck}) + a^j_{ck}\overline{B}^j_{sk}\, q(\chi_j \mathbf{u}^{j0}_{ck}, \chi_j \mathbf{U}^{j1}_{sk}) \\ & + a^j_{sk}\overline{B}^j_{ck}\, q(\chi_j \mathbf{u}^{j0}_{sk}, \chi_j \mathbf{U}^{j1}_{ck}) + a^j_{sk}\overline{B}^j_{sk}\, q(\chi_j \mathbf{u}^{j0}_{sk}, \chi_j \mathbf{U}^{j1}_{sk}) \\ & + b^j_{ck}\overline{A}^j_{ck}\, q(\chi_j \mathbf{u}^{j1}_{ck}, \chi_j \mathbf{U}^{j0}_{ck}) + b^j_{ck}\overline{A}^j_{sk}\, q(\chi_j \mathbf{u}^{j1}_{ck}, \chi_j \mathbf{U}^{j0}_{sk}) \\ & + b^j_{sk}\overline{A}^j_{ck}\, q(\chi_j \mathbf{u}^{j1}_{sk}, \chi_j \mathbf{U}^{j0}_{ck}) + b^j_{sk}\overline{A}^j_{sk}\, q(\chi_j \mathbf{u}^{j1}_{sk}, \chi_j \mathbf{U}^{j0}_{sk}) \Big\}. \end{aligned}$$

Let us calculate the term $q(\chi_j \mathbf{u}^{j0}_{ck}, \chi_j \mathbf{U}^{j1}_{ck})$. We note, firstly, that the cut-off function χ_j restricts all considerations to the cylinder Π^j_+, secondly, that $S(\chi_j \mathbf{u}^{jh}_{\diamond,k})$ and $S^*(\chi_j \mathbf{u}^{jh}_{\diamond,k})$ have compact supports. Applying the Green's formula (12) in the domain $\Omega_L = \{x \in \Omega\colon \text{ if } x \in \Pi^j_+ \text{ then } x^j_3 < L,\ j = 1, \dots, J\}$ we get an additional integral over the cross-section ω^j. Let $\mathbf{n} = (0,0,1)^t$ be the outward normal to $\partial\Omega_L$ on ω^j and $\partial_3 = \partial \setminus \partial x^j_3$. Taking into account (8), (9) and (14), (15) we get

$$\begin{aligned} q(\chi_j \mathbf{u}^{j0}_{ck}, \chi_j \mathbf{U}^{j1}_{ck}) &= (\mathbf{v}^{j0}_{ck}, \mathbf{n}P^{j1}_{ck} - \nu\partial_3 \mathbf{V}^{j1}_{ck})_{\omega^j} + (\mathbf{v}^{j0}_{sk}, \mathbf{n}P^{j1}_{sk} - \nu\partial_3 \mathbf{V}^{j1}_{sk})_{\omega^j} \\ &\quad - (\mathbf{n}p^{j0}_{ck} - \nu\partial_3 \mathbf{v}^{j0}_{ck}, \mathbf{V}^{j1}_{ck})_{\omega^j} - (\mathbf{n}p^{j0}_{sk} - \nu\partial_3 \mathbf{v}^{j0}_{sk}, \mathbf{V}^{j1}_{sk})_{\omega^j} \\ &= -(1, \varphi^j_k)_{\omega^j}. \end{aligned}$$

The rest terms in the Green's formula could be computed in the same way. Finally, we arrive at

$$q(\mathbf{u}_k, \mathbf{U}_k) = \sum_{j=1}^{J} \Big\{ \Big(b_{ck}^j \overline{A}_{ck}^j + b_{sk}^j \overline{A}_{sk}^j - a_{ck}^j \overline{B}_{ck}^j - a_{sk}^j \overline{B}_{sk}^j \Big) (\varphi_k^j, 1)_{\omega^j} + \Big(a_{ck}^j \overline{B}_{sk}^j + b_{sk}^j \overline{A}_{ck}^j - b_{ck}^j \overline{A}_{sk}^j - a_{sk}^j \overline{B}_{ck}^j \Big) (\psi_k^j, 1)_{\omega^j} \Big\}.$$

Now we define operators $\pi_c^0, \pi_s^0, \pi_c^1, \pi_s^1 \colon \mathbb{D}_{\pm\beta}^l W(\Omega) \to \mathbb{C}^J$ (operators $\pi_c^0, \pi_s^0, \pi_c^1, \pi_s^1 \colon \mathbb{D}_{\pm\beta}^l W(\Omega)^* \to \mathbb{C}^J$ are defined in the same way) as follows:

$$\pi_c^0 \mathbf{u} = (a_c^1, a_c^2, \dots, a_c^J)^t, \quad \pi_s^0 \mathbf{u} = (a_s^1, a_s^2, \dots, a_s^J)^t,$$
$$\pi_c^1 \mathbf{u} = (b_c^1, b_c^2, \dots, b_c^J)^t, \quad \pi_s^1 \mathbf{u} = (b_s^1, b_s^2, \dots, b_s^J)^t,$$

where the numbers $a_c^j, a_s^j, b_c^j, b_s^j$ are the coefficients in expansion (11) of the function $\mathbf{u} \in \mathbb{D}_{\pm\beta}^l W(\Omega)$ (in expansion (16) for $\mathbf{U} \in \mathbb{D}_{\pm\beta}^l W(\Omega)^*$). Let

$$c_k^j = \int_{\omega^j} \varphi_k^j \, \mathrm{d}x^{j'}, \quad d_k^j = -\int_{\omega^j} \psi_k^j \, \mathrm{d}x^{j'}, \quad x^{j'} = (x_1^j, x_2^j),$$

and

$$\mathcal{C}_k = \operatorname{diag}\{c_k^1, c_k^2, \dots, c_k^J\}, \quad \mathcal{D}_k = \operatorname{diag}\{d_k^1, d_k^2, \dots, d_k^J\}$$

be the $J \times J$ matrices. Taking into account previous results and notations we get the following formula

$$q(\mathbf{u}_k, \mathbf{U}_k) = \langle \mathcal{C}_k \pi_c^1 \mathbf{u}_k - \mathcal{D}_k \pi_s^1 \mathbf{u}_k, \pi_c^0 \mathbf{U}_k \rangle_J + \langle \mathcal{C}_k \pi_s^1 \mathbf{u}_k + \mathcal{D}_k \pi_c^1 \mathbf{u}_k, \pi_s^0 \mathbf{U}_k \rangle_J - \langle \pi_c^0 \mathbf{u}_k, \mathcal{C}_k \pi_c^1 \mathbf{U}_k + \mathcal{D}_k \pi_s^1 \mathbf{U}_k \rangle_J - \langle \pi_s^0 \mathbf{u}_k, \mathcal{C}_k \pi_s^1 \mathbf{U}_k - \mathcal{D}_k \pi_c^1 \mathbf{U}_k \rangle_J, \quad (17)$$

where $\langle\ ,\ \rangle_J$ stands for a scalar product in $\mathbb{C}^J$. We call (17) the *generalized Green's formula*.

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REZIUMĖ

M. Skujus. Apie Gryno formulę vienam Stokso tipo uždaviniui

Laiko atžvilgiu periodinis Stokso uždavinys begalinių cilindrų sistemoje Furjė eilučių pagalba suvedamas į elipsinių uždavinių seką. Šiems Stokso tipo kraštiniams uždaviniams įvedama apibendrintoji Gryno formulė.

Raktiniai žodžiai: begalinių cilindrų sistema, laiko atžvilgiu periodinis Stokso uždavinys, apibendrintoji Gryno formulė.