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Asymptotic expansions for Yosida approximations of semigroups

Monika VILKIENĖ (MII)
e-mail: monika.vilkiene@ktu.lt

Abstract. In this paper we provide asymptotic expansions for Yosida approximations of contraction semigroups. We also obtain optimal bounds for convergence rate and remainder terms of asymptotic expansions. We use a method introduced in [2] for analysis of errors in Central Limit Theorem and in approximations by accompanying laws. This method was applied in [3] to obtain optimal convergence rates in some approximation formulas for operators and in [10] to obtain asymtotic expansions and optimal error bounds for Euler's approximations of semigroups.

Keywords: semigroups, Yosida approximations, asymptotic expansions, holomorphic semigroups, convergence rate.

1. Introduction and results

In this paper we obtain asymptotic expansions for Yosida approximations of contraction semigroups. At first we provide integro-differential identities

$$S_\lambda(t)x = S(t)x + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \cdots + \frac{a_k}{\lambda^k} + D_k, \tag{1.1}$$

where $S_\lambda(t)$ is Yosida approximation of semigroup $S(t)$ and coefficiens a_m do not depend on λ. We also obtain optimal bounds for convergence rate $\|S(t)x - S_\lambda(t)x\|$ and remainder terms D_k.

To obtain asymptotic expansions we use an approach introduced in [2] for analysis of errors in Central Limit Theorem and in approximations by accompanying laws. Bentkus and Paulauskas in [3] demonstrated that this approach is also useful to get optimal convergence rates in some approximation formulas for operators. In [10] we used this method to obtain asymtotic expansions and optimal error bounds for Euler's approximations of semigroups.

Let X be a Banach space and $L(X)$ be the space of bounded linear operators on X. A function $S\colon \mathbb{R}_+ \mapsto L(X)$ is called a semigroup if it satisfies the semigroup property $S(t+s) = S(t)S(s)$ for all $s, t \geqslant 0$. A semigroup $S(t)$ is called strongly continuous if $S(0) = I$ (I is identity operator on X) and it is continuous function in strong operator topology. If for all $t \geqslant 0$, the norm $\|S(t)\| \leqslant 1$ then $S(t)$ is called a semigroup of contractions.

Let A be a generator of semigroup of contractions. We define the Yosida approximant of A by

$$A_\lambda = \lambda A(\lambda I - A)^{-1}, \tag{1.2}$$

for all $\lambda > 0$. It can be shown (see Lemma 1.3.4 in [8]) that A_λ is the generator of a uniformly continuous semigroup of contractions $S_\lambda(t)$. Furthermore,

$$S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x \quad \text{for } x \in X.$$

We call $S_\lambda(t)$, $\lambda > 0$ Yosida approximations of contraction semigroup $S(t)$.

Assume there exists a positive constant K independent of n, λ and t such that

$$\|tAS(t)\| \leqslant K, \tag{1.3}$$

and

$$(n+1)\|A\lambda^n(\lambda I - A)^{-n-1}\| \leqslant K, \quad n = 0, 1, 2, \dots, \tag{1.4}$$

for all $\lambda > 0$, $t \geqslant 0$.

Bounded holomorphic semigroups satisfy conditions (1.3) and (1.4) by Theorems 2.5.2 and 2.5.5 in [8]. We also prove the following lemma:

LEMMA 1. *Assume that A is a generator of contraction semigroup and there exists a positive constant K independent of n, λ and t such that conditions* (1.3) *and* (1.4) *are satisfied for all $\lambda > 0$, $t \geqslant 0$ and $n = 0, 1, 2, \dots$. Then Yosida approximations satisfy*

$$\|tA_\lambda S_\lambda(t)\| \leqslant K, \tag{1.5}$$

for all $\lambda > 0$ and $t \geqslant 0$.

First we obtain the bound for the convergence rate $\|S(t)x - S_\lambda(t)x\|$.

THEOREM 2. *Assume that semigroup $S(t)$ satisfies conditions* (1.3) *and* (1.5). *Then the following integro-differential identity holds*

$$D_0 = S_\lambda(t)x - S(t)x = \frac{1}{\lambda}\int_0^1 tAA_\lambda S_\lambda((1-\tau)t)S(\tau t)x \,\mathrm{d}\tau, \tag{1.6}$$

for all $\lambda > 0$, and the following inequality holds

$$\|S(t)x - S_\lambda(t)x\| \leqslant \frac{CK\|Ax\|}{\lambda}, \tag{1.7}$$

where C is some absolute positive constant.

Now we provide asymptotic expansions for Yosida approximations. We denote

$$\begin{aligned}
&d_{m,1,1} = 1, \quad m = 1, 2, \dots, \\
&d_{m,m,j} = \frac{1}{m!}, \quad m = 1, 2, \dots, \ j = 1, 2, \dots, m, \\
&d_{m,k,j} = \sum_{i=1}^{j} d_{m-1,k,i}, \quad m=2, 3, \dots, \ k=1, 2, \dots, m-1, \ j=1, 2, \dots, k.
\end{aligned} \tag{1.8}$$

THEOREM 3. *Let $S(t)$ be a differentiable semigroup. Then the coefficients a_m in* (1.1) *are given by*

$$a_m = \sum_{k=1}^{m} d_{m,k,k} t^k A^{m+k} S(t)x, \tag{1.9}$$

and the remainder terms D_m are

$$D_m = D_{m,1} + D_{m,2}, \tag{1.10}$$

where

$$D_{m,1} = \frac{1}{\lambda^{m+1}} \sum_{k=1}^{m} \sum_{j=1}^{k} d_{m,k,j} t^k A^{m+j} A_\lambda^{k+1-j} S(t)x,$$

and

$$D_{m,2} = \frac{1}{\lambda^{m+1}} \int_0^1 \frac{\tau^m}{m!} (tAA_\lambda)^{m+1} S_\lambda\big((1-\tau)t\big) S(\tau t)x \, \mathrm{d}\tau,$$

with coefficients $d_{m,k,j}$ given by (1.5).

For example, the first three coefficients of the expansion are

$$\begin{aligned}
a_1 &= tA^2 S(t)x, \\
a_2 &= tA^3 S(t)x + \frac{t^2 A^4}{2} S(t)x, \\
a_3 &= tA^4 S(t)x + t^2 A^5 S(t)x + \frac{t^3 A^6}{6} S(t)x.
\end{aligned}$$

THEOREM 4. *Assume that semigroup $S(t)$ satisfies conditions* (1.3) *and* (1.5). *Then the remainder terms D_m in* (1.1) *satisfy*

$$\|D_m\| \leqslant \frac{C_m(1 + K^{m+1})\|A^{m+1}x\|}{\lambda^{m+1}}, \quad m = 1, 2, \ldots$$

for $\lambda > 0$ and some positive constant C_m depending only on m.

We note that using the same approach we can obtain the inverse expansions, i.e., expansions of the semigroup $S(t)$ in terms of Yosida approximations $S_\lambda(t)$.

2. Proofs

Proof of Lemma 1. The proof is similar to the proof of Lemma 2.1 in [3]. We have $A_\lambda = \lambda A(\lambda I - A)^{-1} = \lambda^2(\lambda I - A)^{-1} - \lambda I$. Expanding $\mathrm{e}^{t\lambda^2(\lambda I - A)^{-1}}$ into the Taylor series we get

$$tA_\lambda S_\lambda(t) = tA_\lambda e^{tA_\lambda} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1}}{n!} A\lambda^n (\lambda I - A)^{-n-1}.$$

From (1.4) we have $(n+1)\|A\lambda^n(\lambda I - A)^{-n-1}\| \leqslant K$, so that

$$\|tA_\lambda S_\lambda(t)\| \leqslant K e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!} = K(1 - e^{-t\lambda}) \leqslant K,$$

for all $\lambda > 0$ and $t \geqslant 0$.

Proof of Theorem 2. To obtain the convergence rate and asymptotic expansions we use a method introduced by Bentkus in [2]. This method is based on application of Newton-Leibnitz formula along a smooth curve $\gamma(\tau)$, connecting two close objects a and b: $b - a = \gamma(1) - \gamma(0) = \int_0^1 \gamma'(\tau)\, d\tau$. Here we choose γ in this manner

$$\gamma(\tau) = S_\lambda((1-\tau)t)S(\tau t). \tag{2.1}$$

Then $a = S_\lambda(t)$, $b = S(t)$ and

$$\begin{aligned} \gamma'(\tau) &= \big(S_\lambda((1-\tau)t)\big)' S(\tau t) + S_\lambda((1-\tau)t)(S(\tau t))' \\ &= -A_\lambda t S_\lambda((1-\tau)t)S(\tau t) + AtS_\lambda((1-\tau)t)S(\tau t) \\ &= t(A - A_\lambda)\gamma(\tau) = -\frac{1}{\lambda} tAA_\lambda \gamma(\tau). \end{aligned}$$

So, we have

$$D_0 = S_\lambda(t)x - S(t)x = a - b = \frac{1}{\lambda}\int_0^1 tAA_\lambda\gamma(\tau)x\, d\tau. \tag{2.2}$$

Substituting expression (2.1) into (2.2) we obtain (1.6).

Now we obtain the convergence rate $\|D_0\| = \|S_\lambda(t)x - S(t)x\|$ when $S(t)$ is semigroup satisfying conditions (1.3) and (1.4). We denote

$$J_1 = \int_0^{1/2} tAA_\lambda\gamma(\tau)x\, d\tau \quad \text{and} \quad J_2 = \int_{1/2}^1 tAA_\lambda\gamma(\tau)x\, d\tau.$$

Then the convergence rate $\|D_0\| \leqslant \frac{1}{\lambda}(\|J_1\| + \|J_2\|)$. First we estimate $\|J_1\|$. We have

$$\|J_1\| \leqslant \int_0^{1/2} \|tAA_\lambda\gamma(\tau)x\|\, d\tau \leqslant \int_0^{1/2} \frac{\delta_1\delta_2}{1-\tau}\, d\tau,$$

where $\delta_1 = \|AS(\tau t)x\|$ and $\delta_2 = \|(1-\tau)tA_\lambda S_\lambda((1-\tau)t)\|$. Since $S(t)$ is semigroup of contractions, we have $\delta_1 \leqslant \|Ax\|$ and from (1.5) we also have $\delta_2 \leqslant K$. We obtain

$$\|J_1\| \leqslant K\|Ax\| \int_0^{1/2} \frac{1}{1-\tau}\, d\tau = \ln(2)K\|Ax\|. \tag{2.3}$$

Next we estimate $\|J_2\|$. We have

$$\|J_2\| \leqslant \int_{1/2}^{1} \|tAA_\lambda\gamma(\tau)x\|\,\mathrm{d}\tau \leqslant \int_{1/2}^{1} \frac{\delta_3\delta_4}{\tau}\,\mathrm{d}\tau,$$

where $\delta_3 = \|A_\lambda S_\lambda((1-\tau)t)x\|$ and $\delta_4 = \|\tau t A S(\tau t)\|$. By Theorem 1.3.1 in [8] we have that the resolvent of semigroup of contractions satisfies $\|\lambda(\lambda I - A)^{-1}\| \leqslant 1$ for all $\lambda > 0$. It follows that $\|A_\lambda x\| = \|\lambda A(\lambda I - A)^{-1}x\| = \|\lambda(\lambda I - A)^{-1}Ax\| \leqslant \|Ax\|$ and $\delta_3 \leqslant \|Ax\|$. From condition (1.3) we have $\delta_4 \leqslant K$. Then

$$\|J_2\| \leqslant K\|Ax\| \int_{1/2}^{1} \frac{1}{\tau}\,\mathrm{d}\tau = \ln(2)K\|Ax\|, \tag{2.4}$$

and substituting (2.3) and (2.4) into $\|D_0\| \leqslant \frac{1}{\lambda}(\|J_1\| + \|J_2\|)$ we obtain (1.7).

Proof of Theorem 3. From (1.6) we have $S_\lambda(t)x = S(t)x + D_0$ where

$$D_0 = \frac{1}{\lambda}\int_0^1 tAA_\lambda\gamma(\tau)x\,\mathrm{d}\tau.$$

Integrating D_0 by parts we obtain

$$D_0 = \frac{1}{\lambda}tAA_\lambda S(t)x + \frac{1}{\lambda^2}\int_0^1 \tau(tAA_\lambda)^2\gamma(\tau)x\,\mathrm{d}\tau. \tag{2.5}$$

It's easy to prove the following identity $A_\lambda = A + \frac{AA_\lambda}{\lambda}$. Substituting it into the first term of the sum in (2.5) we have

$$D_0 = \frac{tA^2}{\lambda}S(t)x + \frac{tA^2A_\lambda}{\lambda^2}S(t)x + \frac{1}{\lambda^2}\int_0^1 \tau(tAA_\lambda)^2\gamma(\tau)x\,\mathrm{d}\tau = \frac{a_1}{\lambda} + D_1.$$

We proved (1.9) and (1.10) for $m = 1$. Using induction on m we obtain the general result. We omit the proof here.

Proof of Theorem 4. From (1.2) and (1.3) it easily follows that

$$\|D_{m,1}\| \leqslant C_{m,1}K^m\|A^{m+1}x\|/\lambda^{m+1},$$

where $C_{m,1}$ is some positive constant depending only on m. The bound

$$\|D_{m,2}\| \leqslant C_{m,2}K^{m+1}\|A^{m+1}x\|$$

can be obtained in the similar manner as the bound for $\|D_0\|$ in the proof of Theorem 2.

References

1. W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, *Monographs in Mathematics*, **96**, Birkhauser Verlag, Basel (2001).

2. V. Bentkus, A new method for approximations in probability and operator theories, *Liet. mat. rink.*, **43**(4), 444–470 (2003).
3. V. Bentkus, V. Paulauskas, Optimal error estimates in operator-norm approximations of semigroups, *Letters in Mathematical Physics*, **68**(3), 131–138 (2004).
4. E.B. Davies, *One-Parameter Semigroups*, Academic Press, London (1980).
5. K.-J. Engel, R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, *Graduate Texts in Math.*, **194**, Springer Verlag, New York (2000).
6. E. Hille, R.S. Phillips, *Functional Analysis and Semigroups*, *Colloquium Publications*, **31**, American Mathematical Society (AMS) (1957).
7. B.Nagy, J. Zemanek, A resolvent condition implying power boundedness, *Studia Mathematica*, **134**(2), 143–151 (1999).
8. A. Pazy, *Semigroups of Linear Pperators and Applications to Partial Differential Equations*, *Applied Mathematical Sciences*, **44**, Springer-Verlag (1983).
9. R.K. Ritt, A condition that $\lim_{n\to\infty} T^n = 0$, *Proc. Amer. Math. Soc.*, **4**, 898–899 (1953).
10. M. Vilkienė, Another approach to asymptotic expansions for Euler's approximations of semigroups, *Liet. mat. rink.*, **46** (2006).

REZIUMĖ

M. Vilkienė. Pusgrupių Josidos aproksimacijų asimptotiniai skleidiniai

Straipsnyje gauti pusgrupių Josidos aproksimacijų asimptotiniai skleidiniai. Buvo naudojamas metodas, pateiktas Bentkaus (2003) straipsnyje [2].

Raktiniai žodžiai: pusgrupės, Josidos aproksimacijos, asimptotiniai skleidiniai, holomorfinės pusgrupės, konvergavimo greitis.