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# Cut-elimination for knowledge logics with interaction

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**Abstract.** In the article, multimodal logics $K4_n$ and $S4_n$ with the central agent axiom are analysed. The Hilbert type calculi are presented, then the Gentzen type calculi with cut are derived, and the proofs of the cut-elimination theorems are outlined. The work shows that it is possible to construct an analytical Gentzen type calculi for these logics.

*Keywords:* $K4_n$, $S4_n$, interaction axiom, cut elimination, central agent.

## 1. Introduction

In earlier research we analysed multimodal logic $T_n$ enriched with the central agent axiom. In [6] we showed that it is possible to construct an analytical calculus for this logic. In this arcticle we continue our work by presenting the same results for modal logics $K4_n$ and $S4_n$ with the central agent axiom.

We define a propositional formula in a standard recursive way, including operators $\neg$ (negation), $\vee$ (disjunction), $\wedge$ (conjunction), $\supset$ (implication) and a modal operator $K_l$ (meaning "agent l knows"). We say that $l$ can be either $c$, meaning the central agent, or an agent number starting from 1. Capital Latin letters ($A$, $B$, ...) denote any modal logic formula, capital Greek letters ($\Gamma$, $\Delta$, $\Gamma_1$, $\Gamma^*$, $\Gamma^\sharp$) denote a (possibly empty) multiset of modal formulas (the order of the formulas in a multiset does not matter). In the prooftrees we will also use capital Latin letters in square brackets ($[P]$, $[Q]$), which denote some prooftree.

## 2. Hilbert type calculus

First we define a Hilbert type propositional calculus.

DEFINITION 2.1. The propositional Hilbert type calculus (*HPC*) consists of traditional postulati for propositional logic (see [5] or [2]).

Next, we extend the calculus to cover multimodal logics $K4_n$ and $S4_n$.

DEFINITION 2.2. The Hilbert type calculus for logic $K4_n$ ($HK4_n$) consists of all the rules and axioms of *HPC* and:

- axiom $k$: $K_l(A \supset B) \supset (K_l A \supset K_l B)$;
- axiom 4: $K_l A \supset K_l K_l A$;
- rule of necessity: $\frac{A}{K_l A}$, called $K_l$, where $l$ is any agent.

DEFINITION 2.3. The Hilbert type calculus for logic $S4_n$ ($HS4_n$) consists of all the rules and axioms of $HK4_n$ and axiom $t$: $K_l A \supset A$, where $l$ is any agent.

Finally we add the central agent axiom.

DEFINITION 2.4. The Hilbert type calculus for logic $K4_n$ (respectively $S4_n$) with the central agent axiom ($HK4_n^I$, respectively $HS4_n^I$) consists of all the rules and axioms of $HK4_n$ (respectively $HS4_n$) and the axiom $K_i A \supset K_c A$, where $i$ is any agent, except the central one.

## 3. Gentzen type calculus with cut

We use the standard Gentzen type propositional calculus, which can be found in [4], but we do not include structural rules of exchange, because the order of the formulas in antecedent and succedent of a sequent is not important.

DEFINITION 3.1. The Gentzen type propositional calculus (*GPC*) consists of an axiom $A \to A$ and traditional propositional rules, weakening rules, contraction rules and the cut rule.

Similarly we extend *GPC* to include rules for modal logics $K4_n$ and $S4_n$.

DEFINITION 3.2. The Gentzen type calculus for logic $K4_n$ ($GK4_n$) consists of an axiom $A \to A$, all the rules of *GPC* and modal rule for the operator $K_l$:

$$\frac{\Gamma_1, K_l\Gamma_1 \to A}{K_l\Gamma_1, \Gamma_2 \to \Delta, K_l A}\ (\to K_l),$$

where l is any agent.

DEFINITION 3.3. The Gentzen type calculus for logic $S4_n$ ($GS4_n$) consists of an axiom $A \to A$, all the rules of *GPC* and modal rules for the operator $K_l$:

$$\frac{A, K_l A, \Gamma \to \Delta}{K_l A, \Gamma \to \Delta}\ (K_l \to) \qquad \frac{K_l\Gamma_1 \to A}{K_l\Gamma_1, \Gamma_2 \to \Delta, K_l A}\ (\to K_l),$$

where l is any agent.

Then, to model the behaviour of the central agent axiom, we add the rule for the central agent:

DEFINITION 3.4. The Gentzen type calculus with cut for logic $K4_n$ (respectively $S4_n$) with the central agent axiom ($GK4_n^I cut$, respectively $GS4_n^I cut$) consists of an axiom $A \to A$, all the rules of $GK4_n$ (respectively $GS4_n$) and the rule of interaction:

$$\frac{\Gamma \to \Delta, K_i A}{\Gamma \to \Delta, K_c A}\ (\to K_i^c),$$

where $i$ is any agent, except the central one.

In a traditional way (see [3]) we can prove the following:

THEOREM 3.5. *A formula is provable in $GK4^I_n cut$ if and only if it is provable in $HK4^I_n$. A formula is provable in $GS4^I_n cut$ if and only if it is provable in $HS4^I_n$.*

## 4. Gentzen type calculus without cut for $S4_n$

DEFINITION 4.1. The Gentzen type calculus without cut for logic $S4_n$ with the central agent axiom ($GS4^I_n$) consists of:

1) axiom $A \to A$;

2) all the rules of $GS4^I_n cut$, except the cut rule;

3) the rule of interaction:

$$\frac{K_c A, \Gamma \to \Delta}{K_i A, \Gamma \to \Delta}\,(K_i^c \to) \qquad \frac{K_? \Gamma_1 \to A}{K_? \Gamma_1, \Gamma_2 \to \Delta, K_c A}\,(\to K_{?,c}),$$

where $i$ is any agent, except the central one and $K_? \Gamma_1$ consists of formulas, that begins with $K_l$, but $l$ can be different for different formulas in $K_? \Gamma_1$.

The proof of the cut-elimination theorem for logic $S4_n$ is similar to the proof for logic $T_n$ (see [6]). At the begining we replace the cut rule by the mix rule:

DEFINITION 4.2. The Gentzen type calculus with mix for logic $S4_n$ with the central agent axiom ($GS4^I_n mix$) is equivalent to $GS4^I_n cut$, except that the cut rule is replaced by the mix rule:

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi^* \to \Delta^*, \Lambda}\,(\text{mix } A),$$

where $\Pi^*$ and $\Delta^*$ are obtained from $\Pi$ and $\Delta$, respectively, by deleting all the occurrences of formula $A$ (which is called the mix formula).

It is not hard to prove the equivalency:

LEMMA 4.3. *A sequent is provable in $GS4^I_n mix$, if and only if it is provable in $GS4^I_n cut$.*

THEOREM 4.4 (the cut-elimination). *A sequent is provable in $GS4^I_n$, if and only if it is provable in $GS4^I_n mix$.*

*Proof.* The "only if" part is trivial.

For the "if" part we analyse only those proofs, which have only one mix, occurring as their last inference. By induction on applications of the mix rule we can extend this reasoning to all the proofs. Than we define the height, the rank and the grade of the proof, which has only one mix, occurring as its last inference. We say that the height of such a proof will be larger by one than the sum of heights of the proofs of its left

and right sequents. The rank of such a proof is the sum of ranks of the left and right sequents of the mix and the rank of the left (right) sequent is the maximum number of consecutive sequents in all the threads of the proof of that sequent, which contain the mix formula. The grade is the number of logical symbols in the mix formula. However an important difference from the logic $T_n$ case is that $K_i$ adds two to the grade of the mix formula, if $i$ is any agent except the central one. So the grade of $K_i A$ is considered larger than the grade of $K_c A$. For more formal definitions refer to [4].

Another difference from the logic $T_n$ case is that we need one additional lemma, which can be proved easily.

LEMMA 4.5. *If a sequent is provable in $GS4^I_n$, then it is provable in $GS4^I_n$ without an application of the rule $(\to c)$ to the formula $K_i A$.*

And the final difference is that we need three variables for induction. Firstly, we try to lower the grade of the proof. Secondly, we try to keep the grade unchanged and lower the rank of the proof. And finally, if other methods fail, we keep the grade and the rank unchanged and lower the height of the proof. The details of the proof are left to the reader.

The following theorem follows imediately:

THEOREM 4.6. *A sequent is provable in $GS4^I_n cut$, if and only if it is provable in $GS4^I_n$.*

## 5. Gentzen type calculus without cut for $K4_n$

In this case the situation is very simillar to the logic $S4_n$ case.

DEFINITION 5.1. The Gentzen type calculus without cut for logic $K4_n$ with the central agent axiom ($GK4^I_n$) consists of:

1) axiom $A \to A$;

2) all the rules of $GK4^I_n cut$, except the cut rule;

3) the rules of interaction:

$$\frac{K_c A, \Gamma \to \Delta}{K_i A, \Gamma \to \Delta}\,(K_i^c \to) \qquad \frac{\Gamma_1, K_? \Gamma_1 \to A}{K_? \Gamma_1, \Gamma_2 \to \Delta, K_c A}\,(\to K_{?,c}),$$

where $i$ is any agent, except the central one.

The proof of the cut-elimination for logic $K4_n$ is similar to the proof for logic $S4_n$. We will show just one example. Suppose we have the proof, which has only one mix as its last inference. Suppose the rank of the right upper sequent of the mix is larger than 1. Then we must analyse all the possible forms of the mix formula. Suppose the mix formula is of the form $K_c B$. And suppose that the last inference in the right upper sequent of the mix is an application of the $(\to K_{?,c})$ rule. Then we must analyse

all the possible last inferences in the left upper sequent of the mix. Suppose, it is an application of the rule $(\to K_c)$ to the mix formula. Then the proof is of the form:

$$\dfrac{\dfrac{\dfrac{[Q]}{\Gamma_1, K_c\Gamma_1 \to B}}{K_c\Gamma_1, \Gamma_2 \to \Delta, K_cB}(\to K_c) \qquad \dfrac{\dfrac{[R]}{B, K_cB, \Pi_1, K_?\Pi_1 \to C}}{K_cB, K_?\Pi_1, \Pi_2 \to \Lambda, K_cC}(\to K_{?,c})}{K_c\Gamma_1, \Gamma_2, K_?\Pi_1^*, \Pi_2^* \to \Delta^*, \Lambda, K_cC}(\text{mix } K_cB)$$

Now we can first lower the rank of this proof:

$$\dfrac{\dfrac{\dfrac{[Q]}{\Gamma_1, K_c\Gamma_1 \to B}}{K_c\Gamma_1 \to K_cB}(\to K_c) \qquad \dfrac{[R]}{B, K_cB, \Pi_1, K_?\Pi_1 \to C}}{K_c\Gamma_1, B, \Pi_1^*, K_?\Pi_1^* \to C}(\text{mix } K_cB)$$

According to the induction hypothesis we can change this proof to the proof without the mix rule $P_1$. Now we can form the proof with the lower grade:

$$\dfrac{\dfrac{[Q]}{\Gamma_1, K_c\Gamma_1 \to B} \qquad \dfrac{[P_1]}{K_c\Gamma_1, B, \Pi_1^*, K_?\Pi_1^* \to C}}{\Gamma_1, K_c\Gamma_1, K_c\Gamma_1^\sharp, \Pi_1^{*\sharp}, K_?\Pi_1^{*\sharp} \to C}(\text{mix } B)$$

Once again according to the induction hypothesis we can eliminate the mix from this proof to get the proof $P_2$. And than we can change the whole proof to:

$$\dfrac{\dfrac{\dfrac{[P_2]}{\Gamma_1, K_c\Gamma_1, K_c\Gamma_1^\sharp, \Pi_1^{*\sharp}, K_?\Pi_1^{*\sharp} \to C}}{\Gamma_1, K_c\Gamma_1, \Pi_1^\sharp, K_?\Pi_1^* \to C}}{K_c\Gamma_1, \Gamma_2, K_?\Pi_1^*, \Pi_2^* \to \Delta^*, \Lambda, K_cC}(\to K_{?,c})$$

Other cases are left to the reader.

The following theorem follows imediately from the latter proof.

THEOREM 5.2. *A sequent is provable in $GK4_n^I cut$ if and only if it is provable in $GK4_n^I$.*

## 6. Conclusions

In this paper, we showed that it is possible to find an analytical calculi for multimodal logics $K4_n$ and $S4_n$ with central agent axiom. However, further analysis could concentrate on other interaction rules (some of them are presented in [1]). What is more, the central agent axiom could have different properties in different logics (e.g., $K45_n$, $KD45_n$) and consequently add different rules for cut-elimination.

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REZIUMĖ

***J. Andrikonis. Pjūvio pašalinimas sekvenciniuose skaičiavimuose su sąveikos aksioma***

Straipsnyje nagrinėjamos multimodalinės logikos $K4_n$ ir $S4_n$ su centrinio agento sąveikos aksioma. Pristatomi Hilberto tipo skaičiavimai, išvedami Gentzeno tipo skaičia vimai su pjūvio taisykle ir pateikiami pjūvio pašalinimo teoremų įrodymo kontūrai. Darbas demonstruoja, kad šioms logikoms įmanoma sukonstruoti analitinius Gentzeno tipo skaičiavimus.

*Raktiniai žodžiai*: $K4_n$, $S4_n$, sąveikos aksioma, pjūvio pašalinimas, centrinis agentas.