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About the some conditions of the replaceability of the double induction

Livija MALIAUKIENĖ (VPU)
e-mail: maliaukiene@vpu.lt

Abstract. The provability of the axiom of double induction (ADI) with the open induction formula in the additive arithmetic is investigated. The system of additional axioms and theirs provability by ADI is presented.

Keywords: sequential variant of the first order predicate calculus, additive arithmetic, the provability of the axiom of double induction.

In the free variable systems of arithmetic the problem of the replaceability of the axiom of the double induction (ADI) is more interesting in the systems without the restricted difference, as in such systems ADI is definitely stronger than the usual axiom of induction.

In this paper we will be finish doing to investigate of the question (which had begun to study in [1] and [2]) how strong ADI is. In the papers [1] and [2] we was shown, that the axiom of the double induction

$$\forall x A(x, 0) \& \forall y A(0, y) \& \forall xy[A(x, y) \supset A(x', y')] \supset \forall xy A(x, y)$$

can be proved in calculus Z^* (see below) for all open formulae $A(x, y)$ that correspond to the following restriction ($\varkappa$): the formulae of form $mx + q \neq my + t$ (where $m \in \mathbf{N}$; t, q be terms that do not contain variables x and y) can enter in a disjunctive normal form of the formula $A(x, y)$ only in such cases: $t = q$ or $t, q \in \mathbf{N}$. In this paper we will be shown that the restriction ($\varkappa$) can be eliminated.

Below F, G are formulae, $\Gamma, \Delta, Z, \Lambda$ are finite (probably, empty) sequences of formulae. The expression Γ^{α}_{β} shall denote the result change of every occurrence of α in the every formula of Γ for the occurrence β. The expressions $t^{(n)}$, $P^{(n)}t$, $n \cdot t$, where t is an arbitrary term and $n \in \mathbf{N}$, will denote terms accordingly $t^{\overbrace{'' \ldots '}^{n}}$, $\underbrace{P(\ldots(P}_{n}\, t)\ldots)$, $\underbrace{t + \ldots + t}_{n}$.

Let Z be the sequential variant of the first order predicate calculus with the equality, containing the non-logical symbols 0 (zero), $'$ (successor), P (predecessor), $+$ (plus), following postulates:

$P1.\ \Gamma, F, \Delta \to Z, F, \Lambda,$

$P2.\ \Gamma \to Z,\ t = t, \Lambda,$
$A1. \to t' \neq 0,$
$A2. \to P0 = 0,$
$A3. \to Pt' = t,$
$A4. \to t + 0 = t,$
$A5. \to t + s' = (t + s)',$

the rules of substitution

$$\frac{\Gamma^{\alpha}_{d},\ c = d,\ \Delta^{\alpha}_{d} \to Z^{\alpha}_{d}}{\Gamma^{\alpha}_{c},\ c = d,\ \Delta^{\alpha}_{c} \to Z^{\alpha}_{c}}, \qquad \frac{\Gamma^{\alpha}_{d},\ d = c,\ \Delta^{\alpha}_{d} \to Z^{\alpha}_{d}}{\Gamma^{\alpha}_{c},\ d = c,\ \Delta^{\alpha}_{c} \to Z^{\alpha}_{c}},$$

which we shall call S^*-rules, the rules for the logical symbols $\&, \bigvee, \supset, \neg, \sim, \exists, \forall$ and the structural rules (see, e.g. [1]), cut rule

$$\frac{\Gamma \to Z, F; \quad F, \Delta \to \Lambda}{\Gamma, \Delta \to Z, \Lambda}$$

and the axiom of double induction ADI for all open formula $A(x, y)$.

Let Z_1 be the system, obtained from Z by replacement of ADI by the following axioms:

$B1. \to t \neq 0 \supset (Pt)' = t,$
$B2. \to t + s = s + t,$
$B3. \to t + (s + r) = (t + s) + r,$
$B4. \to t + s = t + r \supset s = r,$
$B5. \to nt = ns \supset t = s, \quad n = 2, 3, \ldots,$
$B6^*. \to mt \neq ms + r', \quad r \neq mt,\ r \neq ms$ or $0 < r < m,$
$B7. \to t \neq s \supset \exists r(t + r' = s) \bigvee \exists w(t = s + w').$

LEMMA 1. *The following equivalence is derivable* [1] *in the calculus Z_1:*

$$\underset{Z_1}{\vdash} \quad t \neq p \sim \exists r(t + r' = p) \vee \exists w(t = p + w').$$

LEMMA 2. *Let a, b, z are the terms of the calculus Z_1, $m \in \mathbf{N}$, $z \neq ma$, $z \neq mb$, then*

$$\underset{Z_1}{\vdash} \quad ma \neq mb + z'.$$

THEOREM. *Let $A(x, y)$ is any open formula of the calculus Z_1, then*

$$\underset{Z_1}{\vdash} ADI.$$

[1] The expression $\underset{Z_1}{\vdash} Q$ will denote that the object Q is deducible in the calculus Z_1.

References

1. L. Maliaukienė, The provability of certain sequents in the additive arithmetic, *Liet. mat. rink.*, **35**(4), 518–526 (1995).
2. L. Maliaukienė, The constructive provability of a restricted axiom of double induction in the free variable additive arithmetic, *Liet. mat. rink.*, **37**(1), 61–70 (1997).

REZIUMĖ

L. Maliaukienė. Apie kai kurias dvigubas indukcijos pakeičiamumo sąlygas

Nagrinėjamas dvigubos indukcijos aksiomos (ADI) su bekvantorine indukcine formule įrodomumas adicinėje aritmetikoje, pašalinant [2] straipsnyje taikytą apribojimą. Pateikiama baigtinė aksiomų sistema, kurioje įrodoma ADI bei šių aksiomų įrodomumas su dviguba indukcijos aksioma.

Raktiniai žodžiai: sekvencinis pirmos eilės predikatų skaičiavimas, aditinė aritmetika, dvigubos indukcijos aksiomos įrodomumas.