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Rate of convergence in the transfer theorem for min-scheme

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Abstract. In this paper non-uniform estimate of convergence rate in the min-scheme is obtained. Presented results make the estimates, given in [1] and [2], more precise.

Keywords: min-scheme, limit theorem, rate of convergence.

1. Introduction

Let $\{X_j, j \geqslant 1\}$ be a sequence of independent random variables (r.v.'s) with the common distribution function (d.f.) F.We form r.v.'s

$$W_n = \min(X_1, \dots, X_n), \quad W_{Nn} = \min(X_1, \dots, X_n),$$

where $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued r.v.'s and independent of X_j, $j \geqslant 1$, and linear normalized minima

$$\overline{W}_n = d_n^{-1}(W_n - c_n), \quad \overline{W}_{N_n} = d_n^{-1}(W_{N_n} - c_n), \quad -\infty < c_n < +\infty,\ d_n > 0.$$

Now, let us denote: $V_n(x) = nF(xd_n + c_n)$.

THEOREM 1 ([1]). *Let*

$$\lim_{n\to\infty} V_n(x) = v(x) \tag{1}$$

and

$$\lim_{n\to\infty} P\Big(\frac{N_n}{n} \leqslant x\Big) = \lim_{n\to\infty} A_n(nx) = A(x), \tag{2}$$

then

$$\lim_{n\to\infty} P\big(\overline{W}_{N_n} \leqslant x\big) = \psi(x), \tag{3}$$

where d.f.

$$\psi(x) = 1 - \int_0^\infty \mathrm{e}^{-zv(x)}\,\mathrm{d}A(z).$$

The non-uniform estimate of rate of convergence (deduced from (3)) is presented in this paper. An estimate in the transfer Theorem 1 was obtained in works [2] and [3]. This paper complements the results from Gnedenko [1] and Aksomaitis [2].

2. Main results

A non-uniform estimate of the convergence rate in transfer Theorem 1 is given by the following theorem.

THEOREM 2. *Let conditions* (1) *and* (2) *hold and* $A(+0) = 0$. *Then for any* x, *satisfying conditions* $\frac{V_n(x)}{n} \leqslant \frac{1}{2}$, *the following estimate holds true:*

$$\begin{aligned}\Delta_n(x) &= \left|P\left(\overline{W}_{N_n} \leqslant x\right) - \psi(x)\right| \\ &\leqslant \left(\frac{V_n^2(x)}{n} + \left|\rho_n(x)\right|\right) \int_0^\infty z\delta_n^z(x)\,\mathrm{d}A_n(nz) \\ &\quad + v(x) \int_0^\infty \left|A_n(nz) - A(z)\right| \mathrm{e}^{-zv(x)}\,\mathrm{d}z, \qquad (4)\end{aligned}$$

where

$$\begin{aligned}\rho_n(x) &= v_n(x) - v(x), \\ \delta_n(x) &= \max\left(\left(1 - F(xd_n + c_n)\right)^n, \mathrm{e}^{-v(x)}\right).\end{aligned}$$

Proof. The complete probability formula implies:

$$\begin{aligned}\Delta_n(x) &= \left|\int_0^\infty \left(1 - F(xd_n + c_n)\right)^{nz} \mathrm{d}A_n(nz) - \int_0^\infty \mathrm{e}^{-v(x)z}\,\mathrm{d}A(z)\right| \\ &\leqslant \left|\int_0^\infty \left(1 - F(xd_n + c_n)\right)^{nz} \mathrm{d}A_n(nz) - E\mathrm{e}^{-\frac{v(x)N_n}{n}}\right| \\ &\quad + \left|E\mathrm{e}^{-\frac{v(x)zN_n}{n}} - \int_0^\infty \mathrm{e}^{-v(x)z}\,\mathrm{d}A(z)\right| = \Delta_n^{(1)}(x) + \Delta_n^{(2)}(x). \qquad (5)\end{aligned}$$

We know that ([2])

$$\left|u^\alpha - v^\alpha\right| \leqslant \alpha\left(\max(u, v)\right)^\alpha |\ln u - \ln v|, \quad 0 < u, v \leqslant 1,$$

and

$$\left(1 - \frac{v_n(x)}{n}\right)^n \leqslant \mathrm{e}^{-v_n(x)}, \quad 0 \leqslant v_n(x) \leqslant n.$$

These inequalities lead to

$$\begin{aligned}\left|\left(1 - \frac{v_n(x)}{n}\right)^{nz} - \mathrm{e}^{-v(x)z}\right| &\leqslant z\delta_n^z(x)\left|\ln\left(1 - \frac{u_n(x)}{n}\right)^n + v(x)\right| \\ &\leqslant z\delta_n^z(x)\left(\left|n\ln\left(1 - \frac{v_n(x)}{n}\right) + v_n(x)\right| + \left|v_n(x) - v(x)\right|\right).\end{aligned}$$

Using the inequality

$$\bigl|\ln(1-t)+t\bigr| \leqslant t^2, \quad |t| \leqslant \frac{1}{2},$$

we obtain

$$\left|\Bigl(1-\frac{v_n(x)}{n}\Bigr)^{nz} - \mathrm{e}^{-v(x)z}\right| \leqslant z\delta_n^z\Bigl(\frac{v_n^z(x)}{n} + \bigl|\rho_n(x)\bigr|\Bigr). \tag{6}$$

From (5) and (6) it follows that

$$\Delta_n^{(1)}(x) \leqslant \Bigl(\frac{v_n^z(x)}{n} + \bigl|\rho_n(x)\bigr|\Bigr)\int_0^\infty z\delta_n^z(x)\,\mathrm{d}A_n(nz). \tag{7}$$

Also

$$\begin{aligned}\Delta_n^{(2)}(x) &= \left|\int_0^\infty \mathrm{e}^{-v(x)z}\,\mathrm{d}A_n(nz) - \int_0^\infty \mathrm{e}^{-v(x)z}\,\mathrm{d}A(z)\right| \\ &= \left|\int_0^\infty \mathrm{e}^{-v(x)z}\,\mathrm{d}\bigl(A_n(nz)-A(z)\bigr)\right|.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\Delta_n^{(2)}(x) &= \left|\int_0^\infty \bigl(A_n(nz)-A(z)\bigr)\,\mathrm{d}\mathrm{e}^{-v(x)z}\right| \\ &\leqslant \mathrm{v}(x)\int_0^\infty \bigl|A_n(nz)-A(z)\bigr|\mathrm{e}^{-v(x)z}\,\mathrm{d}z. \end{aligned} \tag{8}$$

The proof of the theorem follows from (5), (7) and (8).

PROPOSITION 1. *If $EN_n < \infty$, then*

$$\Delta_n^{(1)}(x) \leqslant \Bigl(\frac{V_n^2(x)}{n} + \bigl|\rho_n(x)\bigr|\Bigr)\frac{EN_n}{n}.$$

PROPOSITION 2. *If $\rho_n(x) \geqslant 0$, then*

$$\int_0^\infty z\delta_n^z(x)\,\mathrm{d}A_n(nz) \leqslant \int_0^\infty z\mathrm{e}^{-v(x)z}\,\mathrm{d}A_n(nz).$$

3. Examples

Example 1. Let $P(N_n = k) = \frac{1}{n}$, $k = \overline{1,n}$.
We get $\lim_{n\to\infty} A_n(nz) = z$, $0 \leqslant z \leqslant 1$, $|A_n(x) - A(x)| \leqslant \frac{1}{n}$, $\frac{EN_n}{n} = \frac{n+1}{2n}$.

$$\Delta_n^{(1)}(x) \leqslant \Bigl(\frac{V_n^2(x)}{n} + \bigl|\rho_n(x)\bigr|\Bigr)\frac{n+1}{2n}.$$

$$\Delta_n^{(2)}(x) \leqslant \frac{V(x)}{n}\int_0^1 \mathrm{e}^{-zv(x)}\,\mathrm{d}z \leqslant \frac{V(x)}{n}.$$

For instance, $F(x)=\mathrm{e}^x$, $x\leqslant 0$, then $V_n(x)=n\mathrm{e}^{(x-\ln n)}=\mathrm{e}^x$, $\rho_n(x)=0$, and $\Delta_n(x)\leqslant \frac{\mathrm{e}^x}{n}\bigl(1+\frac{\mathrm{e}^x(n+1)}{2n}\bigr)$, provided $2\mathrm{e}^x\leqslant n$.

Example 2. Let $P(N_n=k)=\frac{2k-1}{n^2}$, $k=\overline{1,n}$.

We get $\lim_{n\to\infty}A_n(nz)=A(z)=z^2$, $0\leqslant z\leqslant 1$,

$$\bigl|A_n(x)-A(x)\bigr|\leqslant\frac{2nx+1}{n^2}\leqslant\frac{2n+1}{n^2}\leqslant\frac{3}{n}\cdot\frac{EN_n}{n}=\frac{(n+1)(4n+5)}{6n^2}.$$

$$\Delta_n^{(1)}(x)\leqslant\biggl(\frac{V_n^2(x)}{n}+\bigl|\rho_n(x)\bigr|\biggr)\frac{(n+1)(4n+5)}{6n^2}.$$

$$\Delta_n^{(2)}(x)\leqslant\frac{3V(x)}{n}.$$

If $F(x)=\mathrm{e}^x$, $x\leqslant 0$, then $\Delta_n(x)\leqslant\frac{\mathrm{e}^x}{n}\bigl(\frac{\mathrm{e}^x(n+1)(4n+5)}{6n^2}+3\bigr)$.

References

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REZIUMĖ

A. Aksomaitis. Konvergavimo greičio įvertis minimumų schemos perkėlimo teoremoje

Darbe pateikiamas konvergavimo greičio įvertis stochastinių minimumų schemoje. Rezultatai patikslina Gnedenkų [1] ir A. Aksomaičio [2] gautus rezultatus.

Raktiniai žodžiai: extremes, limit theorem, rate of convergence.