Liet. matem. rink., **47**, *spec. nr.*, 2007, 31–36

A two-dimensional limit discrete theorem for Mellin transforms of the Riemann zeta-function

Violeta BALINSKAITĖ (VU)
e-mail: violeta.balinskaite@mif.vu.lt

Abstract. In the paper two-dimensional limit theorem for the modified Mellin transform of the Riemann zeta-function is obtained.

Keywords: limit theorem, Mellin transform, probability measure, Riemann zeta-function, weak convergence.

Let $\zeta(s)$, $s = \sigma + it$, as usual, denote the Riemann zeta-function. The modified Mellin transforms $\mathcal{Z}_k(s)$ of powers $|\zeta(\frac{1}{2}+it)|^{2k}$, $k \geqslant 0$, are defined, for $\sigma \geqslant \sigma_0(k) > 1$, by

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} x^{-s}\,\mathrm{d}x.$$

In [2] and [3], discrete limit theorems on the complex plane for $\mathcal{Z}_1(s)$ and $\mathcal{Z}_2(s)$, respectively, were proved. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and, for $N \in \mathbb{N} \cup \{0\}$, put

$$\mu_N(\ldots) = \frac{1}{N+1} \sum_{\substack{0 \leqslant m \leqslant N \\ \ldots}} 1,$$

where in place of dots a condition satisfied by m is to be written. Let $h > 0$ be a fixed number.

THEOREM 1 [2]. *Let $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\mu_N(\mathcal{Z}_1(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $N \to \infty$.

THEOREM 2 [3]. *Let $\frac{7}{8} < \sigma < 1$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\mu_N(\mathcal{Z}_2(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $N \to \infty$.

The aim of this note is a two-dimensional limit discrete theorem for the functions $\mathcal{Z}_1(s)$ and $\mathcal{Z}_2(s)$. Define

$$P_{N,\sigma_1,\sigma_2} = \mu_N((\mathcal{Z}_1(\sigma_1 + imh), \mathcal{Z}_2(\sigma_2 + imh)) \in A), A \in \mathcal{B}(\mathbb{C}^2).$$

THEOREM 3. *Suppose that* $\sigma_1 > \frac{1}{2}$ *and* $\frac{7}{8} < \sigma_2 < 1$. *Then on* $(\mathbb{C}^2, \mathcal{B}(\mathbb{C})^2)$ *there exists a probability measure* P_{σ_1,σ_2} *such that the measure* P_{N,σ_1,σ_2} *converges weakly to* P_{σ_1,σ_2} *as* $N \to \infty$.

Let $a > 1$ be a fixed number, for $y \geqslant 1$, $\sigma_0 > \frac{1}{2}$, $v(x, y) = \exp\{-(\frac{x}{y})^{\sigma_0}\}$, and

$$\mathcal{Z}_{k,a,y}(s) = \int_1^a \left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} x^{-s} v(x, y)\,\mathrm{d}x, \quad k = 1, 2.$$

We begin the proof of Theorem 3 with a limit theorem for the vector

$$\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, t) = (\mathcal{Z}_{1,a,y}(\sigma_1 + it), \mathcal{Z}_{2,a,y}(\sigma_2 + it)).$$

For this aim, we apply a limit theorem on the torus

$$\Omega_a = \prod_{u \in [1,a]} \gamma_u,$$

where $\gamma_u = \{s \in \mathbb{C}\colon |s| = 1\} \stackrel{\text{def}}{=} \gamma$ for all $u \in [1, a]$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus Ω_a is a compact topological Abelian group. On $(\Omega_a, \mathcal{B}(\Omega_a))$, define the probability measure

$$Q_{N,a}(A) = \mu_N((u^{imh}\colon u \in [1, a]) \in A).$$

LEMMA 4. *On* $(\Omega_a, \mathcal{B}(\Omega_a))$, *there exists a probability measure* Q_a *such that the probability measure* $Q_{N,a}$ *converges weakly to* Q_a *as* $N \to \infty$.

Proof of the lemma is given in [3], Theorem 5.
Now let

$$P_{N,a,y,\sigma_1,\sigma_2}(A) = \mu_N(\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh) \in A), \quad A \in \mathcal{B}(\mathbb{C}^2).$$

THEOREM 5. *On* $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$, *there exists a probability measure* $P_{a,y,\sigma_1,\sigma_2}$ *such that the probability measure* $P_{N,a,y,\sigma_1,\sigma_2}$ *converges weakly to* $P_{a,y,\sigma_1,\sigma_2}$ *as* $N \to \infty$.

Proof. Define a function $h_{a,y}\colon \Omega_a \to \mathbb{C}^2$ by the formula

$$h_{a,y}(\{y_x\colon x \in [1, a]\}) = \left(\int_1^a \left|\zeta\left(\frac{1}{2} + ix\right)\right|^2 x^{-\sigma_1} v(x, y)\widehat{y_x}^{-1}\,\mathrm{d}x, \right.$$
$$\left. \int_1^a \left|\zeta\left(\frac{1}{2} + ix\right)\right|^4 x^{-\sigma_2} v(x, y)\widehat{y_x}^{-1}\,\mathrm{d}x \right),$$

where

$$\widehat{y}_x = \begin{cases} y_x \text{ if } y_x \text{ is integrable over } [1,a], \\ \text{an arbitrary integrable over } [1,a] \text{ circle function otherwise.} \end{cases}$$

Then the function $h_{a,y}$ is continuous, and

$$\begin{aligned} & h_{a,y}(\{x^{imh}: x \in [1,a]\}) \\ & = \left(\int_1^a \left| \zeta\left(\frac{1}{2}+ix\right) \right|^2 x^{-(\sigma_1+imh)} v(x,y)\,\mathrm{d}x, \int_1^a \left| \zeta\left(\frac{1}{2}+ix\right) \right|^4 x^{-(\sigma_2+imh)} v(x,y)\,\mathrm{d}x \right) \\ & = \underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh). \end{aligned}$$

Therefore, the theorem is a consequence of Lemma 4 and Theorem 5.1 of [1].
By [2], [3], the integrals

$$\mathcal{Z}_{k,y}(s) = \int_1^\infty \left| \zeta\left(\frac{1}{2}+ix\right) \right|^{2k} x^{-s} v(x,y)\,\mathrm{d}x, \quad k = 1, 2,$$

converges absolutely for $\sigma > \frac{1}{2}$ and $\sigma > \frac{7}{8}$, respectively.
Let

$$\underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, t) = (\mathcal{Z}_{1,y}(\sigma_1 + it), \mathcal{Z}_{2,y}(\sigma_2 + it)),$$

and

$$P_{N,y,\sigma_1,\sigma_2}(A) = \mu_N(\underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, mh) \in A), \; A \in \mathcal{B}(\mathbb{C}^2).$$

THEOREM 6. *Let $\sigma_1 > \frac{1}{2}$ and $\frac{7}{8} < \sigma_2 < 1$. Then on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ there exists a probability measure P_{y,σ_1,σ_2} such that the probability measure $P_{N,y,\sigma_1,\sigma_2}$ converges weakly to P_{y,σ_1,σ_2} as $N \to \infty$.*

Proof. Let $M > 0$ be arbitrary number. Then we have that

$$\begin{aligned} & \limsup_{N\to\infty} P_{N,a,y,\sigma_1,\sigma_2}(\{|\underline{z}| \in \mathbb{C}^2: |\underline{z}| > M\}) \\ & = \limsup_{N\to\infty} \mu_N(|\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh)| > M) \\ & \leqslant \limsup_{N\to\infty} \frac{1}{M(N+1)} \sum_{m=0}^{N} |\underline{\mathcal{Z}}_{a,y}(\sigma_1, \sigma_2, mh)| \\ & \leqslant \frac{1}{M} \sup_{a\geqslant 1} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \Big(\sum_{k=1}^{2} |\mathcal{Z}_{k,a,y}(\sigma_k + imh)|^2 \Big)^{\frac{1}{2}} \\ & \leqslant \frac{1}{M} \sup_{a\geqslant 1} \limsup_{N\to\infty} \frac{1}{N+1} \Big(\sum_{m=0}^{N} \sum_{k=1}^{2} |\mathcal{Z}_{k,a,y}(\sigma_k + imh)|^2 \Big)^{\frac{1}{2}} \end{aligned}$$

$$\leqslant \frac{1}{M}\sum_{k=1}^{2}|\mathcal{Z}_{k,y}(\sigma_k)|^2 \leqslant \frac{R}{M}.$$

Hence, taking $M = R\varepsilon^{-1}$, we find that

$$\limsup_{N\to\infty} P_{N,a,y,\sigma_1,\sigma_2}(\{\underline{z}\in\mathbb{C}^2\colon |\underline{z}| > M) \leqslant \varepsilon.$$

Therefore, we obtain that the family of probability measures $\{P_{a,y,\sigma_1,\sigma_2}\colon a \geqslant 1\}$ is tight, and relatively compact. Thus, there exists a subsequence $\{P_{a_1,y,\sigma_1,\sigma_2}\} \subset \{P_{a,y,\sigma_1,\sigma_2}\}$ such that $P_{a_1,y,\sigma_1,\sigma_2}$ converges weakly to some measure P_{y,σ_1,σ_2} as $a_1 \to \infty$.

On a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, define a random variable θ_N by

$$\mathbb{P}(\theta_N = hm) = \frac{1}{N+1}, \quad m = 0, 1, \ldots, N,$$

and put

$$\underline{X}_{N,a,y}(\sigma_1,\sigma_2) = \underline{\mathcal{Z}}_{a,y}(\sigma_1,\sigma_2,\theta_N).$$

Then by Theorem 5,

$$\underline{X}_{N,a,y}(\sigma_1,\sigma_2) \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_{a,y}(\sigma_1,\sigma_2), \tag{1}$$

where $\underline{X}_{a,y}(\sigma_1,\sigma_2)$ is a $\mathbb{C}^2$-valued random element with the distribution $P_{a,y,\sigma_1,\sigma_2}$. Moreover, from above we have that

$$\underline{X}_{a_1,y}(\sigma_1,\sigma_2) \xrightarrow[a_1\to\infty]{\mathcal{D}} P_{y,\sigma_1,\sigma_2}. \tag{2}$$

Denoting by ρ the metric on $\mathbb{C}^2$, we obtain that

$$\lim_{a\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\rho(\mathcal{Z}_{a,y}(\sigma_1,\sigma_2,mh), \mathcal{Z}_y(\sigma_1,\sigma_2,mh))$$
$$\leqslant \lim_{a\to\infty}\limsup_{N\to\infty}\sum_{k=1}^{2}\frac{1}{N+1}\sum_{m=0}^{N}|\mathcal{Z}_{k,a,y}(\sigma_k+imh) - \mathcal{Z}_{k,y}(\sigma_k+imh)| = 0. \tag{3}$$

Now let $\underline{X}_{N,y}(\sigma_1,\sigma_2) = \underline{\mathcal{Z}}_y(\sigma_1,\sigma_2,\theta_N)$. Then, in view of (3), for every $\varepsilon > 0$,

$$\lim_{a\to\infty}\limsup_{N\to\infty}\mathbb{P}(\rho(\underline{X}_{N,a,y}(\sigma_1,\sigma_2), \underline{X}_{N,y}(\sigma_1,\sigma_2)) \geqslant \varepsilon)$$
$$= \lim_{a\to\infty}\limsup_{N\to\infty}\mu_N(\rho(\underline{\mathcal{Z}}_{a,y}(\sigma_1,\sigma_2,mh), \underline{\mathcal{Z}}_y(\sigma_1,\sigma_2,mh)) \geqslant \varepsilon) = 0.$$

This, (1), (2) and Theorem 4.2 of [1] prove the theorem.

Proof of Theorem 3. In view of Theorem 6, in remains to pass from the vector $\underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, mh)$ to

$$\underline{\mathcal{Z}}(\sigma_1, \sigma_2, mh) = (\mathcal{Z}_1(\sigma_1 + imh), \mathcal{Z}_2(\sigma_2 + imh)).$$

In [2] it was proved that, for $\sigma > \frac{1}{2}$,

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} |\mathcal{Z}_1(\sigma + imh) - \mathcal{Z}_{1,y}(\sigma + imh)| = 0,$$

and in [3] it was obtained that, for $\sigma > \frac{7}{8}$,

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} |\mathcal{Z}_2(\sigma + imh) - \mathcal{Z}_{2,y}(\sigma + imh)| = 0.$$

Hence, it follows that

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \rho(\underline{\mathcal{Z}}(\sigma_1, \sigma_2, mh), \underline{\mathcal{Z}}_y(\sigma_1, \sigma_2, mh)) = 0.$$

Therefore, putting

$$\underline{X}_N(\sigma_1, \sigma_2) = \underline{\mathcal{Z}}(\sigma_1, \sigma_2, \theta_N),$$

we derive that, for every $\varepsilon > 0$,

$$\lim_{y\to\infty} \limsup_{N\to\infty} \mathbb{P}(\rho(\underline{X}_{N,y}(\sigma_1, \sigma_2), \underline{X}_N(\sigma_1, \sigma_2)) \geqslant \varepsilon) = 0. \tag{4}$$

By Theorem 6, we have that

$$\underline{X}_{N,y}(\sigma_1, \sigma_2) \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_y(\sigma_1, \sigma_2), \tag{5}$$

where $\underline{X}_y(\sigma_1, \sigma_2)$ is a $\mathbb{C}^2$-valued random element with the distribution P_{y,σ_1,σ_2}. Similarly, as in the proof of Theorem 6, we find that the family of probability measures $\{P_{y,\sigma_1,\sigma_2}\colon y \geqslant 1\}$ is tight. Hence, it is relatively compact. Therefore, there exists a subsequence $\{P_{y_1,\sigma_1,\sigma_2}\} \subset \{P_{y,\sigma_1,\sigma_2}\}$ such that $P_{y_1,\sigma_1,\sigma_2}$ converges weakly to some probability measure P_{σ_1,σ_2} as $y_1 \to \infty$. Hence

$$\underline{X}_{y_1}(\sigma_1, \sigma_2) \xrightarrow[y_1\to\infty]{\mathcal{D}} P_{\sigma_1,\sigma_2}.$$

This, (4), (5) and Theorem 4.2 of [1] again show that

$$\underline{X}_N(\sigma_1, \sigma_2) \xrightarrow[N\to\infty]{\mathcal{D}} P_{\sigma_1,\sigma_2}$$

and the theorem is proved.

References

1. P.Billingsley, *Convergence of Probability Measures*, John Willy and Sons, New York (1968).
2. V. Balinskaitė, A. Laurinčikas, On value distribution of the Mellin transform of the Riemann zeta-function (submited).
3. V. Balinskaitė, A. Laurinčikas, Discrete limit theorems for the Mellin transform of the Riemann zeta-function (submited).

REZIUMĖ

V. Balinskaitė. Dvimatė diskreti ribinė teorema Rymano dzeta funkcijos Melino transformacijoms

Įrodyta dvimatė diskreti ribinė teorema Rymano dzeta funkcijos antrojo ir ketvirtojo laipsnio Melino transformacijoms.