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On the uniqueness of ARCH processes

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Abstract. In this note we prove the uniqueness of the solution to ARCH equations under conditions, which are weaker than in some earlier results.

Keywords: ARCH process, strictly stationary solution..

1. The main result

Let $(\epsilon_k \mid k \in \mathbb{Z})$ be a family of iid nonnegative random variables, $(a_i \mid i \geqslant 1)$ a sequence of nonnegative numbers and $a_0 > 0$. Consider the following system of equations:

$$x_k = \Big(a_0 + \sum_{i=1}^{\infty} a_i x_{k-i}\Big)\epsilon_k, \quad k \in \mathbb{Z}. \tag{1.1}$$

Any strictly stationary nonnegative solution to (1.1), (x_k), is called an *ARCH process*. A solution (x_k) is called *non-anticipative* if, for all k, x_k is independent of ϵ_l, $l > k$.

The most known example of ARCH processes is a sequence (r_k^2), where (r_k) is a so-called *GARCH(p, q) process*, a stationary solution to the equations

$$\begin{aligned} r_k &= \sigma_k \varepsilon_k; \\ \sigma_k^2 &= \delta + \sum_{i=1}^{p} \beta_i \sigma_{k-i}^2 + \sum_{j=1}^{q} \alpha_j r_{k-j}^2, \end{aligned} \tag{1.2}$$

where the ε_k are iid with zero mean, $\delta > 0,\ \beta_i \geqslant 0,\ \alpha_j \geqslant 0$ for all i, j. [4] showed that (r_k^2) satisfies the associated ARCH equations (1.1) with $\epsilon_k = \varepsilon_k^2$, $a_0 = \delta/(1-\beta(1))$ and with coefficients a_i defined by the equality $a(t) = \alpha(t)/(1-\beta(t))$; here $a(t) = \sum_{i=1}^{\infty} a_i t^i$, $\alpha(t) = \sum_{j=1}^{q} \alpha_j t^j$ and $\beta(t) = \sum_{i=1}^{p} \beta_i t^i$.

This paper investigates the question, whether a solution to (1.1) is unique. The main result is the following.

THEOREM 1.1. *Suppose the following conditions are satisfied with some* $q > 1$:

$$\mathrm{E} \log^- \epsilon_0 < \infty, \tag{1.3}$$

$$\sum_{i \geqslant 1} a_i q^i < \infty. \tag{1.4}$$

Then system (1.1) *can have only one strictly stationary solution.*

In the literature, there exist few results concerning uniqueness of ARCH processes. [1] considered GARCH(p,q) processes and proved the uniqueness of the *integrable non-anticipative* solution to (1.2). [3] generalized his results to the general ARCH processes. These results are not comparable with Theorem 1.1: the later does not cover all ARCH processes because of condition (1.4); on the other hand, Theorem 1.1 does not assume integrability of a solution.

[5] considered GARCH(1,1) processes and proved their uniqueness without integrability assumption. [2] generalized his results to the GARCH(p,q) case. Theorem 1.1 generalizes the uniqueness part of Theorem 1.3 of [2], because the coefficients of ARCH equations, associated with (1.2), decay geometrically fast (see [1]).

Finally, [4] proved the uniqueness of an ARCH process under the following assumptions:

(i) a_i decrease, starting from some i_0;

(ii) for some $q > 1$,

$$\sum_{n\geqslant 0} \eta_{kn} q^n < \infty, \tag{1.5}$$

where the η_{kn} are defined by (2.6) below.

In [4], we showed that the convergence radius of the series $\sum_n \eta_{kn} t^n$ does not exceed that of the series $\sum_n a_n t^n$. Therefore condition (1.5) is stronger than (1.4). Moreover, Theorem 1.1 does not require monotonicity of coefficients. On the other hand, in [4] we didn't impose any integrability condition on ϵ_k such as (1.3).

2. The proof

To prove Theorem 1.1, we need two lemmas.

LEMMA 2.1. *Let (ρ_n) be a stationary sequence of quasi-integrable random variables. Then there exists a random variable ξ with values in the extended real line, such that almost surely*

$$\frac{\rho_1 + \cdots + \rho_n}{n} \xrightarrow[n\to\infty]{} \xi. \tag{2.1}$$

Proof. If $\mathrm{E}\,|\rho_1| < \infty$, the lemma follows from the ergodic theorem, see, for example, Shiryaev [7, Chapter V, Theorem 3]. If $\mathrm{E}\,\rho_1^+ < \infty$, $\mathrm{E}\,\rho_1^- = \infty$, it follows from the subbaditive ergodic theorem, applied to the process

$$X_{st} = \rho_{s+1} + \cdots + \rho_t,$$

see [6], Theorem 2. If $\mathrm{E}\,\rho_1^+ = \infty$, $\mathrm{E}\,\rho_1^- < \infty$, the subadditive ergodic theorem should be applied to the process $(-X_{st})$.

LEMMA 2.2. *Suppose, condition* (1.3) *is satisfied and let (x_k) be a stationary solution to* (1.1). *If $a_{i_0} > 0$ for some $i_0 \geqslant 1$, then almost surely*

$$\frac{\log x_{-n}}{n} \to 0. \tag{2.2}$$

Proof. Let (x_k) be a stationary solution to (1.1). By (1.3), $\epsilon_0 > 0$ almost surely. The inequality $x_0 \geqslant a_0\epsilon_0$ then implies that $x_0 > 0$ almost surely. By stationarity, all x_k are positive with probability 1.

For $j \geqslant 1$ define

$$\rho_j = \log \frac{x_{-ji_0}}{x_{-(j-1)i_0}}. \tag{2.3}$$

Clearly, (ρ_j) is a stationary sequence. Moreover, from

$$x_0 \geqslant \epsilon_0\, a_{i_0} x_{-i_0}$$

and (2.3) we get

$$\rho_1 \leqslant \log a_{i_0}^{-1} - \log \epsilon_0;$$

therefore, by (1.3),

$$\mathrm{E}\, \rho_1^+ < \infty.$$

Lemma 2.1 now yields the existence of a random variable ξ, such that almost surely

$$\frac{\rho_1 + \cdots + \rho_j}{j} \xrightarrow[j\to\infty]{} \xi.$$

But $\rho_1 + \cdots + \rho_j = \log x_{-ji_0} - \log x_0$, therefore almost surely

$$\frac{\log x_{-ji_0}}{j} \xrightarrow[j\to\infty]{} \xi. \tag{2.4}$$

On the other hand, $j^{-1}\log x_{-ji_0}$ is distributed identically with $j^{-1}\log x_0$, which tends to 0 almost surely. Therefore

$$\frac{\log x_{-ji_0}}{j} \xrightarrow{P} 0. \tag{2.5}$$

By (2.4)–(2.5), $\xi = 0$, i.e., almost surely

$$\frac{\log x_{-ji_0}}{j} \xrightarrow[j\to\infty]{} 0.$$

By stationarity, for each $d = 0, \ldots, i_0 - 1$,

$$\frac{\log x_{-ji_0-d}}{j} \xrightarrow[j\to\infty]{} 0,$$

which implies that almost surely

$$\frac{\log x_{-ji_0-d}}{ji_0 + d} \xrightarrow[j\to\infty]{} 0.$$

We see that $n^{-1}\log x_{-n}$ tends to 0, as n tends to ∞ along each of the subsequence $n = ji_0 + d$. Therefore, (2.2) holds.

Proof of Theorem 1.1. Denote

$$y_{kn} = a_0\epsilon_k(\eta_{k0} + \eta_{k1} + \cdots + \eta_{kn}), \quad y_k = a_0\epsilon_k \sum_{n\geqslant 0} \eta_{kn},$$

$$z_{kn} = \sum_{i\geqslant n+1} (\eta_{k0}a_i + \eta_{k1}a_{i-1} + \cdots + \eta_{kn}a_{i-n})x_{k-i},$$

where

$$\eta_{kn} = \sum_{i_1+\cdots+i_l=n} a_{i_1}\cdots a_{i_l}\epsilon_{k-i_1}\cdots\epsilon_{k-i_1-\cdots-i_l}. \tag{2.6}$$

In [4] we showed that, for all k and n,

$$x_k = y_{kn} + \epsilon_k z_{kn}. \tag{2.7}$$

Moreover, $y_{kn} \to y_k$, as $n \to \infty$.

All random variables in (2.7) are nonnegative; therefore $x_k \geqslant y_{kn}$ for all n and hence $y_k \leqslant x_k < \infty$, i.e., almost surely

$$\sum_{n\geqslant 0} \eta_{kn} < \infty. \tag{2.8}$$

It is easy to check that the sequence (y_k) is a stationary solution to (1.1). Therefore it remains to prove that $x_k = y_k$ almost surely. To do this, it suffices to show that $z_{kn} \xrightarrow{P} 0$, as $n \to \infty$ (here $\xrightarrow{P}$ stands for the convergence in probability).

If all a_i equal 0, then $z_{kn} = 0$ for all n and there is nothing to prove. Therefore suppose that $a_{i_0} > 0$ for some $i_0 \geqslant 1$. Let $q > 1$ be any number, for which condition (1.4) is satisfied. By Lemma 2.2, almost surely

$$\frac{\log x_{-n}}{n} \to 0 < \log q,$$

hence there exists a random n_0, such that, for all $n \geqslant n_0$, $x_{-n} \leqslant q^n$. Hence, a random variable C, defined by

$$C = \sup_{j\geqslant 1} q^{-j}x_{-j},$$

is almost surely finite.

For all $k \in \mathbb{Z}$, denote

$$C_k = \sup_{j\geqslant 1} q^{-j}x_{k-j}.$$

By stationarity, all C_k are distributed identically with C. Furthermore, for all k and $j \geqslant 1$,

$$x_{k-j} \leqslant C_k q^j. \tag{2.9}$$

Now, by definition of z_{kn} and (2.9),

$$\begin{aligned} z_{kn} &= \sum_{i\geqslant n+1}\sum_{j=0}^{n}\eta_{kj}a_{i-j}x_{k-i} = \sum_{j=0}^{n}\eta_{kj}\sum_{i\geqslant n+1}a_{i-j}x_{k-i} \\ &\leqslant C_{k-n}\sum_{j=0}^{n}\eta_{kj}\sum_{i\geqslant n+1}a_{i-j}q^{i-n} = C_{k-n}\sum_{j=0}^{n}\eta_{kj}q^{j-n}\sum_{i\geqslant n}a_{i-j}q^{i-j} \\ &\leqslant C_{k-n}\sum_{i\geqslant 1}a_i q^i\sum_{j=0}^{n}\eta_{kj}q^{j-n}. \end{aligned}$$

It is well known, that if b_n and c_n are nonnegative numbers, $\sum_n b_n < \infty$ and $c_n \to 0$, then $\sum_{j=0}^{n} b_j c_{n-j} \to 0$. Therefore, by (2.8), almost surely

$$\sum_{j=0}^{n}\eta_{kj}q^{j-n} \xrightarrow[n\to\infty]{} 0.$$

This yields $z_{kn} \xrightarrow{P} 0$, because the sequence C_{k-n} is bounded in probability, as $n \to \infty$.

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REZIUMĖ

V. Kazakevičius, R. Leipus. Apie ARCH procesų vienatį

Šiame darbe įrodome ARCH lygčių sprendinio vienatį esant išpildytoms sąlygoms, kurios yra silpnesnės negu kai kuriuose ankstesniuose darbuose.