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On positive approximations of positive diffusions

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Abstract. For positive diffusions, we construct split-step second-order weak approximations preserving the positivity property. For illustration, we apply the construction to some popular stochastic differential equations such as Verhulst, CIR, and CKLS equations.

Keywords: positive diffusion, weak second-order approximation, CIR equation, CKLS equation.

Introduction. We consider scalar stochastic differential equations (SDEs) of the form

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \quad X_0 = x_0, \tag{1}$$

where B is a standard Brownian motion. Suppose that the coefficients are such that the solution X is positive in the sense that $X_t > 0$ for all $t \geqslant 0$ if $X_0 = x_0 > 0$. In [6], we used the idea of splitting the equation into the deterministic and stochastic parts to construct first-order weak approximations that preserve the (a, b)-invariance property of diffusions. In this note, we develop this idea in order to get *second-order* positive weak approximations for positive diffusions and give several numerical simulation examples.

Second-order weak approximations. For our purposes, the Stratonovich of the equation is more convenient. Thus, instead of (1), we consider equations in the Stratonovich form

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\circ\mathrm{d}B_t, \quad X_0 = x_0 > 0. \tag{2}$$

For Itô equations, the method can be applied by rewriting them in the Stratonovich form.

Recall that the a family of processes $\{X^h,\ h > 0\}$ is said to be a weak approximation of the solution X of order n on the time interval $[0, T]$ if, for all $t \in [0, T]$,

$$\mathbf{E}f(X_t^h) - \mathbf{E}f(X_t) = \mathrm{O}(h^n), \quad h \to 0,$$

for a rather wide class of (test) functions f. In our setting, we take for simplicity this class to be $C_0^\infty(0,\infty)$ (finitary smooth functions on $(0,\infty)$). In this note, we consider weak approximations of Eq. (2) of the form

$$X_0^h = x, \quad X_{(k+1)h}^h = A(X_{kh}^h, h, \Delta B_k), \quad k = 0, 1, 2, \ldots,$$

where the (increment) function $A(x, h, y)$, $(x, h, y) \in (0,\infty)\times[0,\infty)\times\mathbb{R}$, is such that $A(x, 0, 0) = x$, and $\Delta B_k = B_{(k+1)h} - B_{kh}$. The following conditions in terms of an increment function A (together with some technical boundedness conditions) are sufficient for the second-order accuracy of the corresponding approximation [5]:

$$\begin{cases} A'_h(\bar{x}) = \mu(x), \\ A'_y(\bar{x}) = \sigma(x), \\ A''_{yy}(\bar{x}) = \sigma\sigma'(x), \\ \big(2A''_{yh} + A'''_{yyy}\big)(\bar{x}) = (\mu\sigma)'(x) + \sigma(\sigma\sigma')'(x), \\ \big(A''_{hh} + A'''_{yyh} + \frac{1}{4}A''''_{yyyy}\big)(\bar{x}) = \mu\mu'(x) + \frac{1}{2}\mu(\sigma\sigma')'(x) + \frac{1}{2}\sigma(\sigma\mu')'(x) \\ \qquad\qquad + \frac{1}{4}\sigma(\sigma(\sigma\sigma')')'(x), \end{cases} \tag{3}$$

where $\bar{x} = (x, 0, 0)$.

Let now the functions $M(x,t)$, $x > 0$, $t \geqslant 0$, and $S(x,y)$, $x > 0$, $y \in \mathbb{R}$, satisfy the equations $M'_t(x,t) = \mu(M(x,t))$, $M(x,0) = x$, and $S'_y(x,y) = \sigma(S(x,y))$, $S(x,0) = x$, respectively. In other words, the processes $\widetilde{X}_t := M(x,t)$ and $\overline{X}_t := S(x, B_t)$ are (exact) solutions of respectively the *deterministic part*

$$\widetilde{X}_t = x + \int_0^t \mu(\widetilde{X}_s)\,\mathrm{d}s$$

and *stochastic part*

$$\overline{X}_t = x + \int_0^t \sigma(\overline{X}_t) \circ \mathrm{d}B_t$$

of Eq. (2). In [6], we checked that the increment function $A(x,h,y) := S(M(x,h),y)$ defines a first-order weak approximation of the solution of Eq. (2). Similarly, one can check that so does the "adjoint" increment function $\tilde{A}(x,h,y) := M(S(x,y),h)$. Our main result is that:

The average of A and $\tilde{A}$, the increment function

$$A_2(x,h,y) := \frac{1}{2}\big(S(M(x,h),y\big) + M\big(S(x,y),h)\big), \tag{4}$$

satisfies Eqs. (3) *and thus defines a second-order weak approximation of the solution of Eq.* (2).

Examples. 1) Stochastic Verhulst equation:

$$\mathrm{d}X_t = \lambda X_t(1 - X_t)\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t, \quad X_0 = x_0 > 0,$$

or, in the Stratonovich form,

$$\mathrm{d}X_t = X_t(\alpha - \lambda X_t)\,\mathrm{d}t + \sigma X_t \circ \mathrm{d}B_t, \quad X_0 = x_0 > 0,$$

with $\alpha = \lambda - \sigma^2/2$. In this case,

$$S(x,y) = xe^{\sigma y} \text{ and } M(x,h) = \frac{x}{\lambda\alpha^{-1}x + (1 - \lambda\alpha^{-1}x)\exp(-\alpha h)}.$$

For numerical simulation, we have chosen the test function $f(x) = 1/x$. It is motivated by the fact that the process $Y_t := 1/X_t$ satisfies a linear SDE; therefore, we can calculate explicitly the expectation $\mathbf{E}f(X_t) = \exp\{(\sigma^2 - \lambda)t\}\big(\frac{1}{x} + \frac{\lambda}{\sigma^2-\lambda}\big) - \frac{\lambda}{\sigma^2-\lambda}$. In Fig. 1, we plot the exact values of $\mathbf{E}f(X_t)$ together with the approximate expectations obtained by the Euler approximation and first- and second-order approximations defined by the increment functions A and A_2, respectively. Here and in the examples below, the step size $h = 0.5$ and the number of simulated trajectories $n = 2000$.

2) CIR (Cox–Ingersoll–Ross [3]) equation:

$$\mathrm{d}X_t = (a - bX_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}B_t, \quad X_0 = x_0 > 0. \tag{5}$$

Alfonsi [1] constructed and studied several approximations of the CIR equation of weak order one obtained from some implicit schemes that led to analytical formulas. Our approach also leads to analytical formulas. In the Stratonovich form, Eq. (5) becomes

$$\mathrm{d}X_t = (\tilde{a} - bX_t)\,\mathrm{d}t + \sigma\sqrt{X_t} \circ \mathrm{d}B_t, \quad X_0 = x_0 > 0,$$

with $\tilde{a} = a - \frac{\sigma^2}{4}$. For this equation, we easily calculate

$$S(x, y) = \Big(\sqrt{x} + \frac{\sigma}{2}y\Big)^2 \ \text{ and } \ M(x, h) = \frac{\tilde{a}}{b} + \Big(x - \frac{\tilde{a}}{b}\Big)\exp(-bh).$$

For numerical simulation, we take $f(x) = \exp(-2x)$, $a = b = \sigma = 1$, and $x_0 = 0.5$. Then (see, for example, [4], Proposition 6.2.5)

$$\mathbf{E}f(X_t) = \Big(\frac{e^t}{2e^t - 1}\Big)^2 \exp\Big\{-\frac{1}{2e^t - 1}\Big\}.$$

Simulation results are given in Fig. 2.

3) CKLS (Chan–Karolyi–Longstaff–Sanders [2]) equation:

$$\mathrm{d}X_t = (a - bX_t)\,\mathrm{d}t + \sigma X_t^\alpha\,\mathrm{d}B_t, \quad X_0 = x_0 > 0,$$

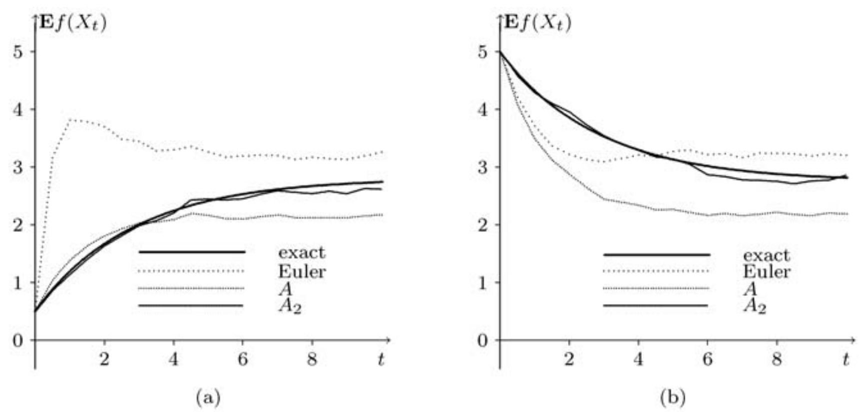

Fig. 1. Approximation of the stochastic Verhulst equation $\mathrm{d}X_t = \lambda X_t(1 - X_t)\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t$, $\lambda = 1$, $\sigma = 0.8$, $f(x) = \frac{1}{x}$; (a) $x_0 = 2$, (b) $x_0 = 0.2$.

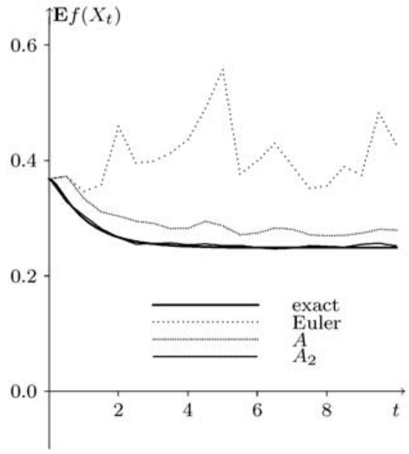

Fig. 2. Approximation of the CIR equation $\mathrm{d}X_t = (a - bX_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}B_t$, $a = b = 1$, $\sigma = 1$, $f(x) = \exp(-2x)$, $x_0 = 0.5$.

with $\alpha \in (1/2; 1)$. As noted by Alfonsi [1], impliciting the drift or diffusion leads to analytical formulas of approximations only in the cases $\alpha = 1/2$ (CIR equation) and $\alpha = 1$ (linear equation).

As before, let us first rewrite the equation in the Stratonovich form:

$$\mathrm{d}X_t = (a - bX_t - cX_t^\beta)\,\mathrm{d}t + \sigma X_t^\alpha \circ \mathrm{d}B_t, \quad X_0 = x_0 > 0,$$

with $\beta = 2\alpha - 1 \in (0, 1)$, $c = \frac{\alpha\sigma^2}{2}$. In this case,

$$S(x, y) = \big(x^{1-\alpha} + (1-\alpha)\sigma y\big)_+^{\frac{1}{1-\alpha}}.$$

However, the deterministic part cannot be solved explicitly. Fortunately, instead of its exact solution, we can take its second-order *positive* approximation

$$M(x, h) = \left(\left(xe^{-bh} + \frac{a}{b}\big(1 - e^{-bh}\big)\right)^{1-\beta} - (1-\beta)ch\right)^{\frac{1}{1-\beta}}, \quad h < h_0,$$

with sufficiently small $h_0 > 0$. The approximation is obtained by splitting the deterministic part $\mathrm{d}X_t = (a - bX_t - cX_t^\beta)\,\mathrm{d}t$ into two equations $\mathrm{d}X_t = -cX_t^\beta\,\mathrm{d}t$ and $\mathrm{d}X_t = (a - bX_t)\,\mathrm{d}t$ with the corresponding solutions $X_t = (x^{1-\beta} - (1-\beta)ct)^{\frac{1}{1-\beta}}$ and $X_t = xe^{-bt} + \frac{a}{b}(1 - e^{-bt})$.

Thus, for the CKLS equation, we also obtain an explicit expression of a second-order weak approximation given by formula (4).

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REZIUMĖ

V. Mackevičius. Teigiamų difuzijų teigiamos aproksimacijos

Teigiamoms difuzijoms sukonstruotos teigiamos antrosios eilės silpnosios aproksimacijos. Naujoji konstrukcija iliustruojama taikymu stochastinėms diferencialinėms Verhulsto, CIR ir CKLS lygtims.